WORKSHOP 1·6

Solutions

Part A

Question: Find the area of a square whose diagonal is of length 2.

Solution There are at least three ways to solve this problem. First, you may use the Pythagorean theorem to figure out the length of a side of the square. Let that length be c, we then have

$$c^2 + c^2 = 2^2 \tag{1}$$
$$2c^2 = 4 \tag{2}$$
$$c^2 = 2 \tag{3}$$
$$c = \sqrt{2}. \tag{4}$$

We then use the formula for the area of a square to get $A = c \cdot c = 2$. The second method involves splitting the square down its diagonal to get two isosceles triangles with a base of length 2 and height of 1. We can then use the triangle area formula to get $A = 2 \cdot \frac{1 \cdot 2}{2} = 2$. Finally we can split the square into four right isosceles triangles of base and height of 1 and we add their area to get $A = 4 \cdot \frac{1 \cdot 1}{2} = 2$.
Note that whichever method we use yields the same result. The methods are resumed in the following diagram.

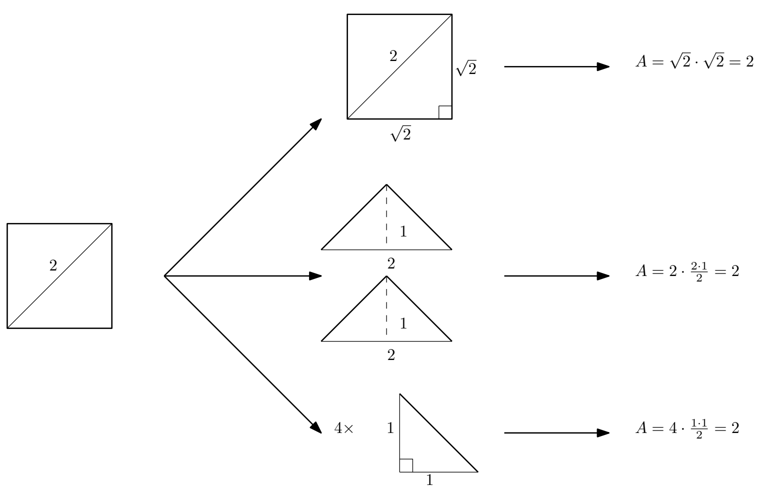

Figure 1: Three methods to find a surface area.

Part B

Question: Find the sum of the first 100 natural numbers.

Solution Here are two ways to solve the solution.
If we pair the numbers such that the first is with the last, the second with the second last, etc. we get the following

$$1 + 100 = 101 \tag{5}$$
$$2 + 99 = 101 \tag{6}$$
$$3 + 98 = 101 \tag{7}$$
$$\cdots 50 + 51 = 101. \tag{8}$$

What we have is 50 pairs each totaling 101. To compute the sum we simply compute 101 times 50 to get 5050.
The second involves a diagram. We wish to build an array by putting one square in the first row, two in the second row, the in the third row and so on. The sum of numbers from 1 to 100 is the number of squares in that staircase diagram. We can then fill the first row with 100 squares, 99 on the second row, 98 on the third row and so on. That second staircase also has a number of square equal to the sum. We have an array of 100 by 101 squares, totaling $100 \cdot 101 = 10100$ squares. Divide this number by two to get the sum of 5050.
Here is an example of such an array if we wished to sum the first 5 numbers.

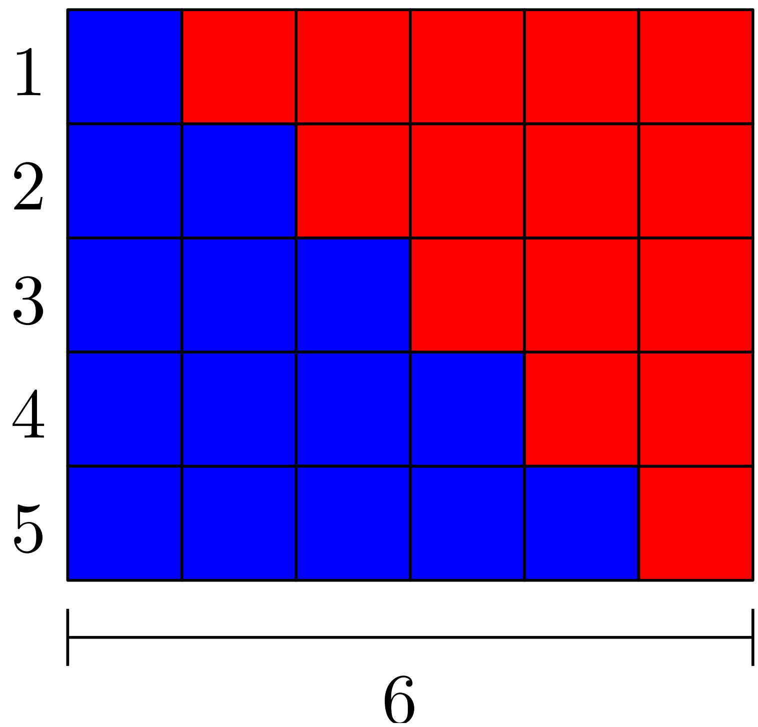

Figure 2: The sum of numbers from 1 to 5 is either the number of blue squares or the number of red squares. To find the total we compute the total in the array $5 \cdot 6 = 30$ and divide by two to get 15.

Note that if you try two different methods and get the same answer both times, something is going right.

Part C

> Question: A hiker sets off on a mountain ascent at 7:00 a.m. in the morning. She reaches the summit at 5:00 p.m. The next morning, she begins her descent at 7:00 a.m., taking the same path back. She reaches the bottom of the mountain at 5:00 p.m. Show, in two different ways, that there is a point on the path that the hiker will cross at exactly the same time of day on both days.

Solution
The key step here is to think about what happens if the hiker is at the same position at the same time. Define $d_1(t)$ and $d_2(t)$ as the distance traveled along the path, counted from the starting position on the first day and second day respectively, we expect that the difference $M(t) = d_1(t) - d_2(t)$ will give, up to possibly a negative sign, the distance along the path between the positions on each day at time t. If it happens that the hiker is at the same position at some time t_0 then that difference should be zero.
To show that the function $M(t)$ Indeed is zero at some point we assume evidently that both $d_1(t)$ and $d_2(t)$ are continuous functions for all times between 7:00 am and 5:00 pm. Therefore their difference $M(t)$ must

also be continuous. Secondly we know that at 7:00 am $M(t)$ has to be negative as on day 1 the distance is zero and on day 2 it is the full distance. Similarly we know that $M(t)$ has to be positive at 5:00 pm because the positions are reversed.

This satisfies the conditions of the Intermediate Value Theorem, so we may use its conclusion, that is there is some time t_0 between 7:00 am and 5:00 pm when $M(t) = 0$ and the hiker is at the same position at the exact same time. Note that we are not required to find that exact time nor that the theorem finds that specific time.

Another way to find the solution is to graph the functions $d_1(t)$ and $d_2(t)$ on the same axes. We know that the first curve must connect the start to the end and the second must connect the end to the start. Then no matter how we climb the mountain on either day there must be at least one intersection point on the graph, denoted by the squares in the diagram below, when the hiker is exactly where he was 24 hours ago.

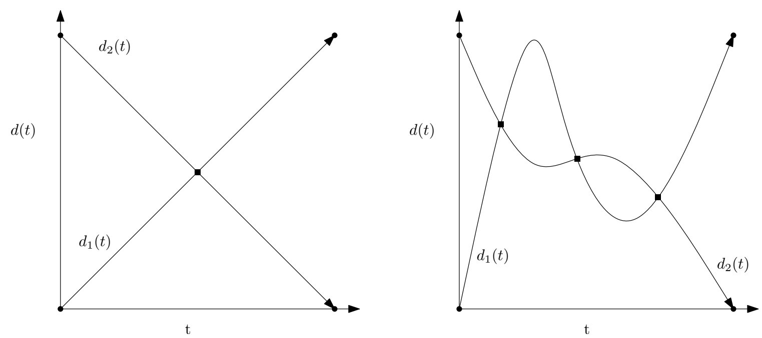

Figure 3: Two possible cases for the mountain hike.