Warm-up Question:
A parking lot charges $3 for the first hour (or part of an hour) and $2 for each succeeding hour (or part), up to a daily maximum of $10. Sketch a graph of the cost of parking at this lot as a function of the time parked there.

Solution:

The left-handed limit at \( t = 2 \) is $5 and the right-handed limit is $7. This means that right before the two hour mark the cost is five dollars and right after the mark it goes up to seven. There is a jump in the cost at that time.

Main Problem:
The federal tax rates for 2016 in Canada are:
- 15% on the first $45,282 of taxable income
- 22% on the next $45,281 of taxable income (on the portion of taxable income between $45,282 and $90,563)
- 26% on the next $49,825 of taxable income (on the portion of taxable income between $90,563 and $140,388)
- 29% on the next $59,612 of taxable income (on the portion of taxable income between $140,388 and $200,000)
- 33% on any income above $200,000.

1. Sketch a graph of a function \( r(x) \) representing the highest tax rate your income would be subject to if your taxable income is \( x \) (the tax rate of your highest taxed bracket). Explain what \( \lim_{x \to 45,282^-} r(x) \) and \( \lim_{x \to 45,282^+} r(x) \) mean and evaluate these limits.
2. Write down a piecewise function representing the amount of tax owed \( t(x) \) as a function of taxable income \( x \). Sketch the graph of the function \( t(x) \). Explain what \( \lim_{x \to 45,282^-} t(x) \) and \( \lim_{x \to 45,282^+} t(x) \) mean and evaluate these limits.

3. Canada’s income tax rates are called progressive because the rate increases as income increases. A flat tax has the same rate for everyone. Suppose income is taxable at a flat rate of 20%, for which income(s) would the flat tax be the same as the current progressive tax? Give a geometrical explanation.

4. (Bonus) Imagine a different tax code that uses the same brackets and rates, but applies the top rate on all the income (for example a person earning $100,000 would be taxed at 26% of $100,000). Roughly sketch what would the tax owed graph look like under that different tax code. Argue in a few sentences about the pros and cons of the different tax codes.

Solution:

1. The graph of maximal rates is at follows:

The left-hand limit gives 0.15 and the right-hand limit is 0.22 and it means that at $45,282 the max rate goes up from 15 % to 22 %.

2. We first solve each part of the piecewise function at a time. For the first case, all the income is taxed at the same rate, therefore the function is:

\[
t_1(x) = 0.15x.
\]

For the second case, we first tax the first $43,953 at 15 % and the rest at 22 %:

\[
t_2(x) = 0.15 \cdot 45,282 + 0.22(x - 45,282).
\]

For the third case we tax for the first cut and the second cut at their respective rate and tax the remainder at 26 %:

\[
t_3(x) = 0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26(x - 90,563),
\]

then onto the next bracket:

\[
t_4(x) = 0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26 \cdot 49,825 + 0.29(x - 140,388).
\]

Finally we do the same for the last case:

\[
t_5(x) = 0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26 \cdot 49,825 + 0.29 \cdot 59,612 + 0.33(x - 200,000).
\]
Putting everything together gives:

\[
t(x) = \begin{cases} 
0.15x & x \leq 45,282 \\
0.15 \cdot 45,282 + 0.22(x - 45,282) & 45,282 < x \leq 90,563 \\
0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26(x - 90,563) & 90,563 < x \leq 140,388 \\
0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26 \cdot 49,825 + 0.29(x - 140,388) & 140,388 < x \leq 200,000 \\
0.15 \cdot 45,282 + 0.22 \cdot 45,281 + 0.26 \cdot 49,825 + 0.29 \cdot 59,612 + 0.33(x - 200,000) & x \geq 200,000 
\end{cases}
\]

and its plot looks like the following, where the vertical axis is the taxes owed:

Both sided limits should give \(0.15 \cdot 45,282 = 6792.30\) this means that the amount owed goes continuously according to the income, meaning that there is no jump in the amount of tax a person would pay as its income increases. This prevents some situations around the end of each brackets when someone with a higher income would pay less than a lower income one due to a jump down or a person having to pay significantly more taxes just for earning a few more dollars (see part 4 for an example).

3. A flat tax rate would give a tax function as follows:

\[f(x) = 0.20x\]

To find the income we put the two tax functions equal and we solve for \(x\). We expect to be in the third case by then. We may verify this by comparing the progressive and flat tax amounts at the different boundaries and check when one passes the other.

\[
f(x) = t_3(x)
\]

\[
0.20x = 0.15 \cdot 45,282 + 0.22 \cdot 45,282 + 0.26(x - 90,563)
\]

\[
-0.06x = 0.15 \cdot 45,282 + 0.22 \cdot 45,281 - 0.26 \cdot 90,563
\]

\[
x = \frac{1}{0.06} \cdot (0.15 \cdot 45,282 + 0.22 \cdot 45,282 - 0.26 \cdot 90,563) = 113,204
\]

Geometrically, that is when the curves of the two functions’ graphs intersect, as shown in the graph of the two taxes.
4. Under the new tax code the line segments of the progressive code would become disconnected as the following figure shows

The function is now disconnected and that would mean that there is a possible incentive to earn less money in order to significantly pay less taxes and end up with more, which is a serious disadvantage from the progressive or single flat rate codes.