WORKSHOP 1·13

Solutions

Warm-up Problem

Question: Your village depends for its food on a nearby fishpond. One day the wind blows a water lily seed into the pond, and the lily begins to grow. It doubles its size every day, and its growth is such that it will cover the entire pond (and smother the fish population) in 45 days. You are away on vacation, and the villagers depend on your vigilance and intelligence to handle their affairs. Without you present they will take action about that lily only on regular work days, and only when the lily covers half the pond or more. How many days after it first starts growing will the lily cover half the pond? The day it reaches that size happens to be a holiday....what do you do?

Solution

We know that the lily is as large as the pond in 45 days and it doubles in size each day. This means that every previous day the plant was half the size of the current day, therefore in 44 days the lily will be half the size of the pond. Hopefully you notified the villagers before leaving otherwise they will have to find a new pond or repopulate the current one with new fish.

Worked Exampled While solving warm up problem does not require any algebra, exponential growth and decay problems often require the manipulation of equations containing exponentials and logarithms. Let's review the basic properties and definitions that will help you work with exponential models.

1) By definition of logarithmic function as the inverse of the exponential function, if $\ln x = y$ then $e^y = x$ and $e^{\ln x} = x$. The exponential function has nice properties (laws of exponents) [list properties]. Similarly, the logarithmic function has analogous properties [list properties].

Let us use these definition and properties to solve this problem:

Example: Assume that the number of bacteria at time t (in minutes) follows an exponential growth model $P(t) = Ce^{kt}$, where C and k are constant. Suppose the count in the bacteria culture was 300 after 15 minutes and 1800 after 40 minutes. When will the culture contain 3000 bacteria?

Solution:

We know the following two values for $P(t)$:

$$P(15) = Ce^{15k} = 300$$
$$P(40) = Ce^{40k} = 1800.$$

What we need to do is to solve for C and k. One way to do this is to divide the two equations:

$$\begin{aligned}
\frac{P(15)}{P(40)} = \frac{Ce^{15k}}{Ce40k} &= \frac{300}{1800} \\
e^{15k-40k} &= \frac{1}{6} \\
e^{-25k} &= \frac{1}{6} \\
-25k &= \ln\frac{1}{6} = -\ln 6 \\
k &= \frac{\ln 6}{25} \approx 0.072.
\end{aligned}$$

We can then substitute the value of k into one of the two equations to obtain C:

$$\begin{aligned} C &= 300e^{-15k} \\ &= 300e^{-\frac{15}{25}(\ln 6)} \approx 102.38. \end{aligned}$$

To find when the culture will contain 3000 bacteria we need to solve for time:

$$\begin{aligned} 3000 &= Ce^{kt} \\ \ln \frac{3000}{C} &= kt \\ t &= \frac{\ln 3000 - \ln C}{k} \\ t &= \frac{\ln 3000 - 300e^{-\frac{15}{25}(\ln 6)}}{\frac{\ln 6}{25}} \\ t &\approx \frac{\ln 3000 - \ln 102.38}{0.072} \approx 46.91 \text{ minutes} \end{aligned}$$

Main Problem

Two cities, Growth and Decay, have populations that are respectively increasing and decreasing, at (different) rates that are proportional to the respective current population (that is, we can model the population size as an exponential function). Growth's population is now 3 million and was 2 million 10 years ago, and Decay's population is now 5 million and was 7 million 10 years ago.

Question: How many years from now will Growth's population equal 4 million?

Solution:

Let $G(t)$ be the function of the population in millions of Growth in millions with time. We know the current population and the population from ten years ago:

$$G(0) = 3, \qquad G(-10) = 2.$$

Since the populations change according to the exponential function, we know that $G(t) = C_G e^{k_g t}$ and we need to solve for C_G and k_G:

$$\begin{aligned} G(0) &= C_G e^{k_G \cdot 0} = 3 \\ G(-10) &= C_G e^{k_G \cdot (-10)} = 2. \end{aligned}$$

From the first equation we obtain $C_G = 3$, which we can use in the second equation to solve for k_G:

$$\begin{aligned} 3e^{-10k_G} &= 2 \\ -10k_G &= \ln \frac{2}{3} = \ln 2 - \ln 3 \\ k_G &= \frac{\ln 3 - \ln 2}{10} \approx 0.0405. \end{aligned}$$

We then have a formula for $G(t)$. To find when Growth's population is 4 million we need to solve for time:

$$\begin{aligned} G(t) = C_G e^{k_G t} &= 4 \\ k_G t &= \ln \frac{4}{3} = \ln 4 - \ln 3 \\ t &= \frac{\ln 4 - \ln 3}{\frac{\ln 3 - \ln 2}{10}} \\ t &= 10 \frac{\ln 4 - \ln 3}{\ln 3 - \ln 2} \approx 7.10 \text{ years} \end{aligned}$$

Question: How many years from now will Decay have half the population of Growth?

Solution:
To find this we need to find the formula to Decay's population first. Let $D(t) = C_D e^{k_D t}$ be the formula we need to solve the following system of unknowns:

$$\begin{aligned} D(0) = C_D &= 5 \\ D(-10) = C_D e^{-10k_D} &= 7 \\ \frac{5}{7} &= \frac{C_D}{C_D e^{-10k_D}} \\ 10k_D &= \ln 5 - \ln 7 \\ k_D &= \frac{\ln 5 - \ln 7}{10} \approx -0.0336. \end{aligned}$$

We then need to solve for time when Decay's population is one half of Growth's population:

$$\begin{aligned} D(t) &= \frac{G(t)}{2} \\ C_D e^{k_D t} &= \frac{C_G}{2} e^{k_G t} \\ \frac{e^{k_D t}}{e^{k_G t}} &= \frac{C_G}{2C_D} \\ e^{(k_D - k_G)t} &= \frac{C_G}{2C_D} \\ (k_D - k_G)t &= \ln \frac{C_G}{2C_D} = \ln C_G - \ln C_D - \ln 2 \\ t &= \frac{\ln C_G - \ln C_D - \ln 2}{k_D - k_G} \approx 16.23 \text{ years} \end{aligned}$$

Question: At what rate is the population changing for the two towns at that moment?

Solution:
We can find the derivative of $G(t)$ and $D(t)$ using the chain rule:

$$G'(t) = (C_G e^{k_G t})' = C_G e^{k_G t} \cdot k_G D'(t) \qquad = k_D \cdot C_D e^{k_D t}.$$

Therefore the rates of changes for the two populations is obtained by multiplying the population at that time with the value of k. For Growth we get

$$G'(t) = k_G \cdot C_G e^{k_G t} \approx 0.0405 \cdot 3e^{0.0405 \cdot 16.23} \approx 0.235 \text{ million per year.}$$

Similarly, for Decay we get

$$D'(t) = k_D \cdot C_D e^{k_D t} \approx -0.0336 \cdot 5e^{-0.0336 \cdot 16.23} \approx -0.0975 \text{ million per year.}$$

Bonus Question 1: Find a formula for the rate of change of the combined population of Growth and Decay at time t. Show that the rate goes from negative to positive. What does that mean in terms of population?

Solution
We know the derivative for $G(t)$ and $D(t)$ from the previous questions and we also know that the derivative of a sum is the sum of the derivatives. Let $P(t)$ be the total population, then we know that

$$\begin{aligned} P(t) &= C_G e^{k_G t} + C_D e^{k_D t} \\ P'(t) &= k_G \cdot C_G e^{k_G t} + l_D \cdot C_D e^{k_D t}. \end{aligned}$$

Let's try to find a time when the derivative is zero if it exists. This will give us the time of transition from an increasing to a decreasing population or the other way around:

$$\begin{aligned}
0 &= k_G \cdot C_G e^{k_G t} + k_D \cdot C_D e^{k_D t} \\
k_G \cdot C_G e^{k_G t} &= -k_D \cdot C_D e^{k_D t} \\
\frac{e^{k_G t}}{e^{k_D t}} &= \frac{-l_D \cdot C_D}{k_G \cdot C_G} \\
e^{(k_G - k_D)t} &= \frac{-k_D \cdot C_D}{k_G \cdot C_G} \\
(k_G - k_D)t &= \ln \frac{-k_D \cdot C_D}{k_G \cdot C_G} = \ln -k_D + \ln C_D - lnk_G - \ln C_G \\
t &= \frac{\ln \frac{-k_D \cdot C_D}{k_G \cdot C_G} = \ln -k_D + \ln C_D - lnk_G - \ln C_G}{k_G - k_D} \approx 4.37 \text{ years.}
\end{aligned}$$

Thus after about 4.37 years there are as many people leaving Decay as there are going to Growth. To find whether the function is increasing before and after that time we check the derivative at a time before and after.
One easy choice it $t = 0$. in this case the derivative is

$$P'(0) = k_G \cdot G(0) + k_D \cdot D(0) \approx 0.0405 \cdot 3 - 0.0336 \cdot 5 \approx -0.0466 \text{ million per year}$$

Another possible choice would be the time obtained in the second part of the question $t \approx 16.23$. We know that at that time the population of Growth is twice that of Decay:

$$\begin{aligned}
P'(16.23) &= k_G \cdot G(16.23) + k_D \cdot D(16.23) && \approx k_G \cdot 2D(16.23) + k_D \cdot D(16.23) \\
&= (2k_G + k_D)D(16.23) \approx 0.0474D(16.23) > 0.
\end{aligned}$$

So we know that the total population is decreasing at $t = 0$, is increasing at $t = 16.23$ and stays the same at about $t = 4.37$, therefore the combined population of the two cities goes from decreasing to increasing and the change occurs at about 4.37 years.

Bonus Question 2: A third city, Grecay, has a lot of seasonal workers, such that its peak in the summer (around July 1st) it has the same population as Growth and at its lowest in the winter (around January 1st) it has only half of the population. Using the formula you obtained previously for Growth's population, try to find a possible formula for Grecay's population after t years. Note that the population has to remain continuous.
Hint: Try a population function of the form $P(t) = G(t) \cdot h(t)$, where $G(t)$ is Growth's population and $h(t)$ is a periodic function, like sine or cosine, might be useful here.

Solution:
Considering that the annual minimum of population occurs at around n years, for the nth year and the population peak at $n + 0.5$ years, we want $h(t)$ to be a periodic function that is equal to 0.5 when $t = 0 + n$ and equal to 1 when $t = 0.5 + n$.
We also know that $-\cos(t)$ is minimal for $t = 0 + 2\pi m$ and maximal for $t = \pi + 2\pi m$ for any integer m. If we change the variable into $-\cos(2\pi t)$ we have maximums at half integers and minimums at integers.
All that is left is to make $h(t)$ have the correct value at its maximum and minimum. To do this we will assume that $h(t) = A - B\cos(2\pi t)$. Doing this only translate the graph up by A and streches the amplitude by a factor of B. Therefore the maxima and minima stay where they are. To find the values of A and B we

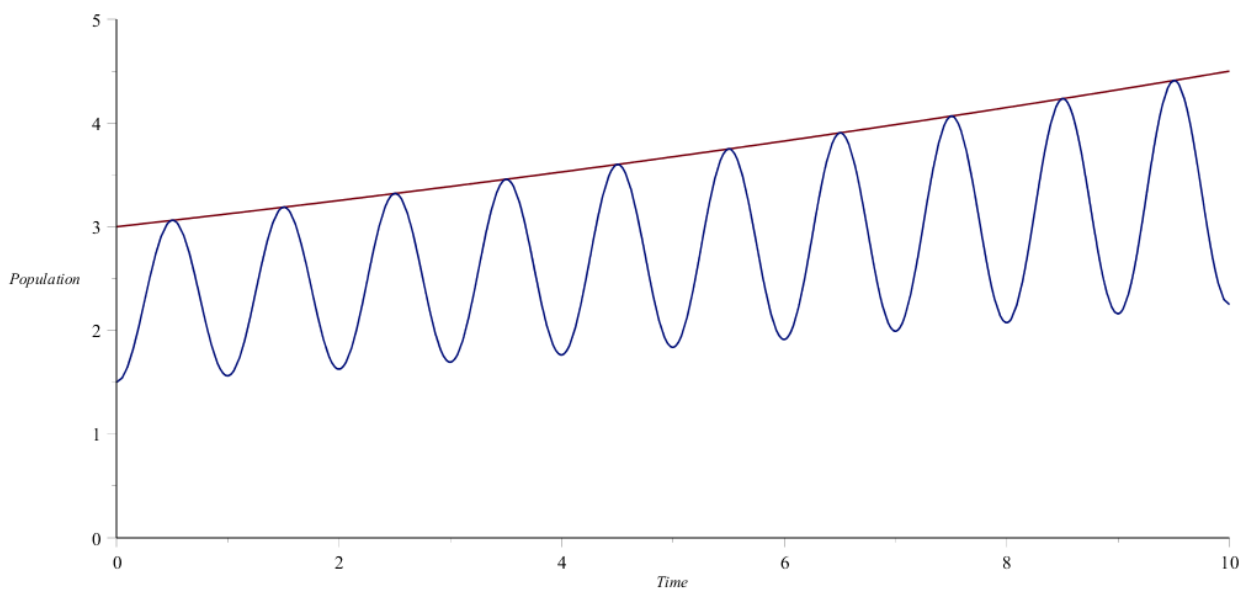
just need to solve the following two equations:

$$\begin{aligned} h(0) &= A - B\cos(0 \cdot 2\pi) = A - B = \frac{1}{2} \\ h(0.5) &= A - B\cos(0.5 \cdot 2\pi) = A + B = 1 \\ 2A &= \frac{3}{2} \quad \text{(adding the previous two lines.)} \\ A &= \frac{3}{4} \\ B &= 1 - \frac{3}{4} = \frac{1}{4}. \end{aligned}$$

Therefore $h(t) = \frac{3}{4} - \frac{1}{4}\cos(2\pi t)$ and the population of Decay is therefore

$$P(t) = \left[\frac{3}{4} - \frac{1}{4}\cos(2\pi t)\right] C_G e^{k_G t}.$$

We have a plot of both Grecay's and Growth's populations in the first 10 years from now.



Note that the maximum and minimum of $P(t)$ are not exactly at $t = 0$ and $t = 0.5$. The reason for this can be seen if we take the derivative of $P(t)$, using the product rule:

$$P'(t) = G'(t)h(t) + G(t)h'(t) = C_G \cdot k_G e^{k_G t}\left(\frac{3}{4} - \frac{1}{4}\cos(2\pi t)\right) + C_G e^{k_G t} \cdot 2\pi \sin(2\pi t).$$

Where we expected the maxima and minima to happen, when t is an integer or half integer the second term of the derivative will be zero, but not the first term. However, that second term is generally more important than the first one and if we solve when the derivative is zero we get a difference of less than one day, so it is close enough for our purposes.
Note that there could have been other choices for $h(t)$. For example we could have chosen a periodic piecewise function with an increasing line from 0.5 to 1 and then decreasing from 1 to 0.5. The important part is that $h(t)$ has the correct two values at integer and half-integer values.