WORKSHOP 1·12

Handout

Warm-up Problem

Question: How many times do you need to fold this paper to reduce it to 25% (1/4) of its initial size? How about 6.25% (1/16)? Can you come up with an equation that relates the number of times you fold the paper with the number of sub-squares produced by the fold?

Solution:
Each time you fold the paper you double the amount of sub-rectangles each of which is half the size of the previous ones. As a result the surface area of the paper after n folds is

$$A(n) = A_0 \cdot \left(\frac{1}{2}\right)^n,$$

where A_0 is the initial surface area. Computing the first few values of $A(n)$ we see that folding twice gives $A(2) = \frac{1}{4} \cdot A_0$ and folding four times yields $A(4) = \frac{1}{16} \cdot A_0$, the required sizes. One would have to fold ten times to have a piece smaller than one thousandth of the original ($A(10) = \frac{1}{1024} A_0$).

Worked Exampled

Example: NASA has received intel on a zombie infection on earth. To avoid human extinction NASA designs a spacecraft to take a group of people to the international space station (where they will meet up with Sandra Bullock and George Clooney). The height of the spacecraft above earth, in kilometres, can be described using the function: $H = 5^t$, where t is in sec after launching. At the same time as the spacecraft is launched I start walking across the room. Where will the spacecraft be by the time I have reached the other end of the room (∼12m)? Assume an average walking speed of 1.5m/s. How fast will the spacecraft be traveling at this time?

Solution:
We can find easily enough that it takes 8 seconds to walk through the room (12 m / 1.5 m/s = 8 s). We can then compute the eight power of 5:

$$H = 5^8 = 390625 \text{ km}.$$

This is very far, farther than the distance between the Earth and the Moon. We can easily find the speed V at that time:

$$V = H' = 5^8 \cdot \ln 5 \approx 628687 \text{ km/s},$$

which is twice the speed of light. The moral of the story here is that the exponential function grows very quickly.

Main Problem

Somehow, as usual, the evacuation was only available to the elite and you were left behind on earth with most of its population. This doesn't sound so bad at first because the zombie infection doesn't seem to be spreading as fast as you would have expected.

After a few months, scientists discover that the fraction of the abandoned population that has been infected can be determined in a precise way as a function of time. They release their findings to the public which state that the fraction $f(t)$ of the remaining population that has been infected t years after the initial outburst is given by the following function:

$$f(t) = \frac{1}{1 + 9e^{-t}}$$

Question: According to the model, what fraction of the population was infected by the initial outburst?

Solution:
The initial infected population fraction is simply the infected population fraction at $t = 0$ years:

$$f(0) = \frac{1}{1 + 9e^0} = \frac{1}{10},$$

so the initial outbreak affects one tenth of the population.

Question: According to the model, after how many years will half of the population be infected?

Solution:
For this problem we want to find time t when the fraction of infected, given by $f(t)$ is one half, that is,

$$f(t) = \frac{1}{2}.$$

We can then use algebra to find the value of t

$$\begin{aligned} f(t) = \frac{1}{1 + 9e^{-t}} &= \frac{1}{2} \\ 1 + 9e^{-t} &= 2 \\ 9e^{-t} &= 1 \ (*) \\ e^{-t} &= \frac{1}{9} \\ e^{t} &= 9 \\ t &= \ln(9) \approx 2.20 \text{ years.} \end{aligned}$$

Question: According to the model, what is the rate at which the fraction of the population consisting of zombies is increasing at this point in time?

Solution:
The rate at which the population is being infected is given by the derivative of $f(t)$ evaluated at $t = \ln(9)$. We first need to find the derivative first:

$$\begin{aligned} f'(t) &= \frac{(1)'(1 + 9e^{-t}) - (1)(1 + 9e^{-t})'}{(1 + 9e^{-t})^2} \text{ (Quotient Rule)} \\ f'(t) &= \frac{0 - 9(e^{-t})'}{(1 + 9e^{-t})^2} \\ f'(t) &= \frac{9e^{-t}}{(1 + 9e^{-t})^2} \text{ (Chain Rule)} \\ f'(\ln(9)) &= \frac{9e^{-\ln(9)}}{(1 + 9e^{-\ln(9)})^2} \\ f'(\ln(9)) &= \frac{1}{(1 + 1)^2} = \frac{1}{4}, \end{aligned}$$

so at that time the infected grow at a rate of a quarter of the population per year. Note that to find the derivative of e^{-t} we used the chain rule with outside function $U(x) = e^x$ and inside function $V(t) = -t$, but we could have used the quotient rule on $e^{-t} = \frac{1}{e^t}$ and we would have obtained the exact same derivative of $-e^{-t}$.
Also note that we used the result in (*) to find the value of $9e^{-t} = 9e^{-\ln(9)} = 1$.

Follow-up Question: According to the model, when will the last human become a zombie?

Solution:
because we know that the exponential function is always strictly positive, there is no time t_0 where $e^{-t_0} = 0$. However inputting higher values of time will yield a fraction that is ever closer to 1. Therefore will there always be one human that cannot be infected?
There is no single answer to this problem, because it depends on some assumptions. The equation $f(t)$ is only a model of the outbreak, and by all means it is reasonable to assume that at some point everyone will be infected. Because of this there will be a threshold fraction when we consider everyone to be infected. Whether it is $f(t) = 0.99$ or 0.9999. Note that the choice of threshold will change the time. The question is now how to choose a proper fraction?
We have to consider that we have a total population here. For the sake of this example we reduce the problem to a town of $50,000$ inhabitants. The function $f(t)$ will give some fraction of infected humans that won't divide exactly into on integer, for example one third of the population yields about 16666.67 Does that mean that there are 16666 infected? Or 16667? There is one human two thirds infected?
This is where we could choose our threshold. For example we could say that if 49999.5 humans are infected, then everyone is infected, then we can use $\frac{49999.5}{50000}$ as our fraction and solve for t:

$$\begin{aligned}
\frac{49999.5}{50000} = \frac{99999}{100000} &= \frac{1}{1+9e^{-t}} \\
1+9e^{-t} &= \frac{100000}{99999} \\
9e^{-t} &= \frac{100000}{99999} - 1 = \frac{1}{99999} \\
e^{-t} &= \frac{1}{9 \cdot 99999} = \frac{1}{899991} \\
e^{t} &= 899991 \\
t &= \ln(899991) \approx 13.7 \text{ Years.}
\end{aligned}$$

So it would take about 13.7 years until the last human becomes infected according to our assumption. One other assumption would be to have the last human be considered infected if it is any fraction infected.