WORKSHOP 1·11

Handout

Warm-up Problem

Consider two functions f and g such that

- $f'(x) < 0$ for $x > 0$ and $f'(x) > 0$ for $x < 0$,
- both $g(x)$ and $g'(x)$ are negative for all values of x.

Find at what rate the composite function $f(g(x))$ is changing when $x = 2$. Can the rate of change of $f(g(x))$ be positive for some value of x?

Worked Example

A small group of mathematicians are starting a soup business and they have big plans. As the company grows, they could hire more and more employees with time and the more employees they hire the more soup they can produce. The model uses the following functions for the employment and production functions E(t) and P(t):

$$E(t) = at^2 + bt + 5, \quad P(x) = c\sqrt{x},$$

where t is measured in time and x is the number of employees. a, b and c are positive constants that can be adjusted. Find an expression for the rate of change in production with time for this model.

Main Problem

While designing their future production plant, one of the entrepreneurs thinks a large tank in which the various ingredients would be mixed. As is often the case with mathematician's designs, the shape of the tank is spherical, as it requires the least amount of material to hold the maximum amount of soup.
If a spherical tank of radius 4 metres is filled up to h metres with water, then the volume of water in the tank is given by the following formula:

$$V = \frac{\pi}{3}h^2(12 - h)$$

Questions:

(a) At what rate is the volume of soup in the tank changing with respect to the height of the soup when h = 1 m? What are the units on this quantity?

(b) Now suppose that the height of soup in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the soup at time t is given by the rule $h(t) = \sin(\pi t) + 1$, where t is measured in hours (and h is still measured in metres). At what rate is the height of the soup changing with respect to time at the instant $t = 2$?

(c) Continuing under the assumptions in (b), at what instantaneous rate is the volume of soup in the tank changing with respect to time at the instant $t = 2$?

(d) What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

(e) Suppose now that the height of soup in the tank is given by the piecewise function

$$h(t) = \begin{cases} \sin(\pi t) + 1, & t \leq 2 \\ \frac{1}{2}\cos(\pi t) + \frac{1}{2}, & t > 2 \end{cases}$$

Is the volume function with time $V(h(t))$ continuous at $t = 2$?