WORKSHOP 1·11

Handout

Warm-up Problem

Consider two functions f and g such that

- $f'(x) < 0$ **for** $x > 0$ **and** $f'(x) > 0$ **for** $x < 0$**,**
- **both** $g(x)$ **and** $g'(x)$ **are negative for all values of** x**.**

Find at what rate the composite function $f(g(x))$ is changing when $x = 2$. Can the rate of change of $f(g(x))$ be positive for some value of x?

Main Problem

A small group of mathematicians are starting a soup business and they have big plans.

While designing their future production plant, one of the entrepreneurs imagines a large tank in which the various ingredients would be mixed. As is often the case with mathematician's designs the shape of the tank is spherical, as it requires the least amount of material to hold the maximum amount of soup.
If a spherical tank of radius 4 metres is filled up to h metres with water, then the volume of water in the tank is given by the following formula:

$$V = \frac{\pi}{3} h^2 (12 - h)$$

Questions:

(a) **At what rate is the volume of soup in the tank changing with respect to the height of the soup when h = 1 m? What are the units on this quantity?**

(b) **Now suppose that the height of soup in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the soup at time t is given by the rule $h(t) = \sin(\pi t) + 1$, where t is measured in hours (and h is still measured in metres). At what rate is the height of the soup changing with respect to time at the instant $t = 2$?**

(c) **Continuing under the assumptions in (b), at what rate is the volume of soup in the tank changing with respect to time at the instant $t = 2$?**

(d) **What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.**

(e) **Suppose now that the height of soup in the tank is given by the piecewise function**

$$h(t) = \begin{cases} \sin(\pi t) + 1, & 0 \le t \le 2 \\ \frac{1}{2}\cos(\pi t) + \frac{1}{2}, & t > 2 \end{cases}$$

Is the volume function with time $V(h(t))$ continuous at $t = 2$? Is it differentiable at $t = 2$?