WORKSHOP 1·10

Solution

Warm-up Question:

1. Evaluate $\sin(-\pi)$, $\cos(\frac{5}{2}\pi)$, $\tan(\frac{3\pi}{4})$.

   Solution:
   $\sin(-\pi) = 0$, $\cos(\frac{5}{2}\pi) = 0$, $\tan(\frac{3\pi}{4}) = \frac{\sin(\frac{3\pi}{4})}{\cos(\frac{3\pi}{4})} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1$

2. Solve the following equations for x in $[0, 2\pi]$: $\sin(x) = 1$, $\cos(x) = -1$.

   Solution:
   For $\sin(x) = 1$, $x = \frac{\pi}{2}$ in $[0, 2\pi]$. For $\cos(x) = -1$, $x = -\pi$ in $[0, 2\pi]$.

Main Problem:

Part A: Solve the following problem that appeared in the MATH 110 2016 December exam.

Consider the function $f(x) = \dfrac{\sin(x)}{1 - \cos(x)}$.

1. Find the domain of f.

   Solution:
   $f(x)$ is defined for all x in $(-\infty, \infty)$ except where $1 - \cos(x) = 0$, which occurs when $\cos(x) = 1$. $\cos(x) = 1$ when x is an integer multiple of 2π. So, the domain of $f(x)$ is $(-\infty, \infty) \setminus \{\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots\}$.

2. Find all x-intercepts of the curve $y = f(x)$ in the interval $[-2\pi, 2\pi]$.

   Solution:
   x-intercepts occur where $f(x) = 0$, which occurs when $\sin(x) = 0$. $\sin(x) = 0$ when $x = -2\pi, -\pi, 0, \pi, 2\pi$, for x in $[-2\pi, 2\pi]$. However, $f(x)$ is not defined at $x = -2\pi, 0, 2\pi$. So, the x-intercepts of the curve $y = f(x)$ occur only at $x = -\pi, \pi$ in the interval $[-2\pi, 2\pi]$.

3. Find $f'(x)$. Simplify your answer as much as possible.

   Solution: Fro the quotient rule,

   $$f'(x) = \frac{\big(1 - \cos(x)\big)\cos(x) - \sin(x)\sin(x)}{\big(1 - \cos(x)\big)^2}.$$

   From the identity $\sin(x)^2 + \cos(x)^2 = 1$ and some algebra,

   $$f'(x) = \frac{\cos(x) - 1}{\big(1 - \cos(x)\big)^2} = \frac{-\big(1 - \cos(x)\big)}{\big(1 - \cos(x)\big)^2} = \frac{-1}{1 - \cos(x)}.$$

4. Explain why there are no points on the curve $y = f(x)$ where the tangent line has slope 1.

   Solution: If the tangent line to the curve $y = f(x)$ has slope 1, then

   $$f'(x) = \frac{-1}{1 - \cos(x)} = 1$$

   for some x, which implies that $-1 = 1 - \cos(x)$. $-1 = 1 - \cos(x)$ if $\cos(x) = 2$, but the range of $\cos(x)$ is $[-1, 1]$. So, there is no x where $f'(x) = 1$, and thus no points on the curve $y = f(x)$ where the tangent line has slope 1.