Solutions

1. $\sum \frac{n}{n^2+1}$ diverges.

by comparison $n^2+1 < n^2+n^2$

$$\frac{1}{n^2+1} > \frac{1}{2n^2}$$

$$\frac{n}{n^2+1} > \frac{n}{2n^2} = \frac{1}{2n}$$ which diverges

(compare with $\lim_{n\to\infty} \frac{\frac{1}{2n}}{\frac{1}{n}} = \frac{1}{2}$ ← harmonic series)

Alternatively, by limit comparison test

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^2+1}\cdot n = 1 \neq 0$$ so the series

behaves like $\sum \frac{1}{n}$.

2. $\sum \frac{1}{100n+2000}$ diverges

by comparison $100n+2000 < 2100n$

$$\frac{1}{100n+2000} > \frac{1}{2100n}$$ diverges

or limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{100n+2000}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{100n+2000} = \frac{1}{100} \neq 0$$

So series behaves like $\sum \frac{1}{n}$.

Alternatively, integral test also applies

$$\int_n^\infty \frac{1}{100x+2000}\,dx = \lim_{t\to\infty} \frac{1}{100}\ln|100x+2000|\Big|_n^t = \lim_{t\to\infty} \frac{1}{100}\ln(100t+2000) - \frac{\ln n}{100}$$

3. $\sum \frac{1+n}{2+n}$ diverges because $\lim_{n\to\infty} a_n \neq 0$, in fact

$$\lim_{n\to\infty} \frac{1+n}{2+n} = 1 \neq 0.$$

4. $\sum \frac{1}{\pi^n+5}$ converges by comparison test

$$\pi^n + 5 > \pi^n$$

$$\frac{1}{\pi^5+5} < \frac{1}{\pi^n}$$ which converges because geom. series and $\frac{1}{\pi} < 1$

5. $\sum \frac{1}{\ln(n)+n}$ diverges by comparison

$$\ln n + n < n + n \text{ for large } n$$

$$\frac{1}{\ln n + n} > \frac{1}{2n}$$ which diverges.

Alternatively, by limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n + n} = 1$$

$$\hookrightarrow \lim_{x\to\infty} \frac{x}{\ln x + x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+1} = 1$$

6. $\sum_{n=0} 3^{2+n} 2^{1-3n} = \sum 3^2 \cdot 2 \cdot 3^n \cdot 2^{-3n} = \sum_{n=0} 18\left(\frac{3}{8}\right)^n$ converges because geom. serie and $\frac{3}{8} < 1$

7. $\sum \frac{1+\sin n}{n^2}$ converges by comparison

$$1 + \sin(n) \leq 2$$

$$\frac{1+\sin(n)}{n^2} \leq \frac{2}{n^2}$$ which converges (p-series, $p=2$)

8. $\sum \frac{\sqrt{n}}{n^2+n+1}$ converges by comparison test

$$n^2+n+1 > n^2$$

$$\frac{1}{n^2+n+1} < \frac{1}{n^2}$$

$$\frac{\sqrt{n}}{n^2+n+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$ p-series, $p=\frac{3}{2}$ converges.

so series converges.

9. $1+\frac{1}{\ln 2}+\frac{1}{3}+\frac{1}{\ln 4}+\frac{1}{5}+\frac{1}{\ln 6}+\dots$ diverges.

by comparison

$$1+\frac{1}{\ln 2}+\frac{1}{3}+\frac{1}{\ln 4}+\frac{1}{5}+\dots > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$$

because $\ln 2 < 2$, $\ln 3 < 3$, etc. harmonic series diverges.

10. $\sum_{n=1} \frac{n\cos(n\pi)}{2^n}$ alternating series with $a_n = \frac{n}{2^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2^n} = 0$$

$f(x) = \frac{x}{2^x}$ decreasing for $x > \frac{1}{\ln 2}$

so series converges.

also could test for absolute convergence.

$\sum \left| \frac{n\cos(n\pi)}{2^n} \right| = \sum \frac{n}{2^n}$ which converges by ratio test,

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$

11. diverges because $\lim_{n\to\infty} a_n \neq 0$

12. diverges by comparison to $\sum \frac{1}{2n}$

13. converges by ratio test.

14 converges by alternating series test.

15 diverges because $\lim_{n\to\infty} a_n \neq 0$

16. converges by integral test or comparison to $\sum \frac{1}{n^3}$

17. diverges by comparison to $\sum \frac{1}{n}$

18. converges by comparison to $\sum \frac{1}{n^2} = \sum \frac{n}{\sqrt{n^6}}$

19. converges absolutely by comparing to $\sum \frac{1}{n^4}$

20. diverges by comparison to $\sum \frac{1}{2n}$

(recall $n^{\frac{1}{n}} \to 1$ for $n \to \infty$, so $n^{\frac{1}{n}} < 2 \Rightarrow \frac{1}{n \cdot n^{1/n}} > \frac{1}{2n}$)

21. converges because geometric series and $|\cos(1)| < 1$