SCIENCE ONE, MATHEMATICS - HOMEWORK #2.3

Due on Wednesday, April 4th by 10AM.

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1. (a) Find the values of p for which the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$ diverges, converges conditionally, and converges absolutely.

   (b) Find the values of p for which the alternating series $\sum_{n=2}^{\infty}(-1)^{n-1} \frac{(\ln n)^p}{n^2}$ diverges, converges conditionally, and converges absolutely.

2. In the next problems you will be working with some simple examples of fractals. A fractal is a mathematical set that displays a self-similarity property, that is, it exhibits a repeating pattern that displays at every scale.

   (a) The Cantor set, named after the German mathematician Georg Cantor (1845-1918), is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $(1/3, 2/3)$. That leaves the two intervals $[0, 1/3]$ and $[2/3, 1]$ and we remove the open middle interval third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in the original interval $[0, 1]$ after all those intervals have been removed.

   (a) Give examples of some numbers in the Cantor set.

   (b) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set is not an empty set.

   (b) The Sierpinski carpet is a two-dimensional counterpart of the Cantor set. It is constructed by removing the centre one-ninth of a square of side 1, then removing the centres of the eight smaller remaining squares, and so on. A visualization of the Sierpinski carpet is available on Wikipedia (`https://en.wikipedia.org/wiki/Sierpinski_carpet`).

   (a) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.

   (c) In this problem you will construct Helga von Koch's snowflake curve. Start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then remove the middle part. Step 2 is to repeat step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely. Animated visualisations of Helga von Koch's snowflake curve (and many variants of the same curve) are available on Wikipedia (`https://en.wikipedia.org/wiki/Koch_snowflake`).

   Here you will show that the snowflake curve is a curve of infinite length that encloses a region of finite area.

   (a) Sketch the three curves that are obtained after the first two steps in the construction process described above.

   (b) Let p_n be the perimeter of the polygon obtained after step n of the construction. Find a formula for p_n. *Hint: First find a formula for the number of sides of the polygon, then find the length of each side.*

(b) Show that $p_n \to \infty$ as $n \to \infty$.

(c) Sum an infinite series to find the area enclosed by the snowflake curve. *Hint: First compute the area of the original triangle, then add the area of each smaller triangle.*

3. Determine whether the electric field produced at the origin by the following arrangements of infinitely many point charges is finite or infinite:

   (a) equal charges arranged along a half line at $x = 1$, $x = 2$, $x = 3$, and so on.

   (b) charge q at $x = 1$, charge $2q$ at $x = 2$, charge $3q$ at $x = 3$, and so on.

   (c) charge q at $x = 1$, charge $-2q$ at $x = 2$, charge $3q$ at $x = 3$, charge $-4q$ at $x = 4$, and so on.

   (d) BONUS equal charges in a two-dimensional array, one charge at each point (j, k) for each $j, k = 1, 2, 3, ...$ (except at the origin).