SCIENCE ONE, MATHEMATICS - HOMEWORK #2.3

Due on Wednesday, March 14 at 10AM.

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1. The Gamma function $\Gamma(x)$ is a continuous function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

   (a) Compute $\Gamma(1)$.

   (b) Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

   (c) Show that $\Gamma(n+1) = n!$ when n is a positive integer.

   (d) Using the fact that $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, compute $\Gamma(\frac{1}{2})$. Then find $\Gamma(\frac{3}{2})$, $\Gamma(\frac{5}{2})$, and so forth.

   Remark: In view of (c), $\Gamma(x+1)$ is often written $x!$ and regarded as a real-valued extension of the factorial function. Some scientific calculators with the factorial function $n!$ built in actually calculate the gamma function rather than just the integral factorial. Check whether your calculator does this by asking it for 0.5!. If you get an error message, it's not using the gamma function.

2. *The Exponential Eiffel Tower*: Completed in 1889, the Eiffel Tower in Paris is one of the most recognized landmarks in the world. It rises 300 metres from a 100-metre square base to a 10-metre square observation deck. Surprisingly, the project's chief engineer, Gustave Eiffel, left no detailed structural analysis that explained the design of the tower; it is believed that the tower was not designed according to an accurate mathematical model, but simply based on empirical observations and practical experience of its designer. In particular, Eiffel was concerned with the effects of wind on a free-standing structure of this size and he developed a construction technique that relied on the idea that the tangents to the skyline profile of the tower at some height h (representing the direction of the tension between the construction elements) must intersect at the centre of mass of that part of the tower that stands above the height h. It turns out that exponential functions have the same property (a mathematical model for the profile of the Eiffel Tower based on this property is discussed in Weidman and Pinelis, *Comptes Rendus Mecanique*, Vol 332, no. 7, 2004).

   Consider the region R bounded by the curves $f(x) = e^{-cx}$ and $g(x) = -e^{-cx}$ on an interval $[a, b]$, where $b > a \geq 0$ and $c > 0$.

   (a) Sketch the region R with $a = 0$ and find its centre of mass. (If you interpret x as height above the ground, the two curves approximate the profile of the Eiffel Tower).

   (b) Show that the tangent lines to the curves $y = f(x)$ and $y = g(x)$ at $x = 0$ intersect at the centre of mass of R if $b \to \infty$. Note that this property holds for any value of $a > 0$.

   (c) (BONUS POINT) Show that the same property holds for any pair of functions $y = \pm Ae^{-cx}$ where A is any positive real number.

   Remark: Of course the Eiffel Tower is not infinitely high, so the condition $b \to \infty$ is not satisfied. A more accurate model for the profile of the tower would require a piecewise function with two exponentials in order to model the two main sections of the tower.

3. In the hydrogen atom ground state (1s orbital), the probability density function for the distance $r \geq 0$ (in units of Bohr radii) of the electron to the nucleus is

$$f(r) = 4r^2e^{-2r}.$$

   (a) Find the mean distance of the electron to the nucleus.

(b) By using a Taylor expansion, estimate the (small) radius δ about the nucleus within which there is a $\frac{1}{48}$ probability of finding the electron.

4. A clepsydra, or water clock, is a container with a small hole in the bottom through which water can flow. Water clocks were used by ancient Egyptians, Greeks, Romans and Chinese to measure time by observing the change in the height of water in the container. The device is calibrated for measuring time by placing markings on the container corresponding to water levels at *equally spaced times.*

   Suppose the container of a water clock is obtained by revolving around the y-axis the graph of a continuous function $x = f(y)$ defined on an interval $[0, b]$, where b is some positive real number. Let V denote the volume of water in the container and h the height of the water level at time t. Find a formula for the function $f(y)$ that would result in a water clock with *equally spaced* markings on the container, i.e. with markings that are an *equal vertical distance* from each other.

   You may use the fact that the rate of change of the volume of water V in a container with a hole of area A at the bottom of the container is proportional to the square root of the water height h (Torricelli's Law),

$$\frac{dV}{dt} = -A\sqrt{2gh},$$

   where g is the acceleration due to gravity.