Science One - Mathematics December Exam, December 9th, 2016
Duration: 150 minutes
*This test has **11 questions** on **13 pages**, for a total of 77 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: __________ Last Name: __________

Student-No: __________ Section: __________

Signature: __________

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	6	6	6	6	8	12	6	9	6	6	6	77
Score:												

6 marks

1. Determine whether each of the following statements is **true or false**. If it is true, provide justification. If it is false, provide justification or a counterexample.

   (a) True/False. An equation of the tangent line to the curve $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

   (b) True/False. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

   (c) True/False. Below is a table of values for a function f.

t	-2	-1	0	1	2	3	4
$f(t)$	4	-1	3	8	1	1	-5

   Then f must be zero at at least three points in the interval $(-2, 4)$.

6 marks

2. (a) Evaluate $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if $f(x) = e^{\tan x}$.

(b) An object of mass M (in kg) moving on a line is subject to a force (in N) $F = -kx - cv$, where x is its position (in m), v is its velocity (in m/s), and k and c are constants (with units of N/m and Ns/m respectively). If the initial position is $x(0) = 2$, the initial velocity is $v(0) = 4$, and $M = 1$, $k = 2$, $c = 1$ (and assuming Newton's second law holds), estimate the object's position 0.5 s later using a Taylor polynomial of degree 2.

(c) Suppose a quantity $q(t)$ decreases in time according to the model $q' = -kq$, where k is a positive constant. Suppose it takes 3 days for a certain amount of q to decrease to a third of the original amount. Find k.

$\boxed{\text{6 marks}}$ 3. A rectangular block with square base is being slowly squeezed in such a way that its volume remains constant, while its height h is decreasing and the edge x of its base is increasing (assume it maintains its box-like shape as it's being squeezed). What is the height of the block (in terms of x) at the instant when both h and x are changing at the same rate (in absolute value)?

6 marks 4. (a) Suppose a curve is described by the equation $x^2 + (y+1)e^y = 5$, find the slopes of the tangent lines to the curve at the points where $y = 0$.

(b) If $y(x) = (\ln(x))^{e^{(x^2)}}$, find $y'(x)$ (for $x > 1$).

8 marks

5. Evaluate the following limits if they exist. If a limit does not exist, determine whether or not it is (positive or negative) infinity. Justify your answers.

(a) $\displaystyle\lim_{x\to\pi/2} \frac{1 - sinx}{1 - cos2x}$

(b) $\displaystyle\lim_{t\to 0^+} t \ln t$

(c) $\displaystyle\lim_{u\to\infty} [\ln(2 + u^2) - \ln(1 + u^2)]$

(d) $\displaystyle\lim_{x\to 0} x^2 \cos\left(\frac{20\pi}{x}\right)$

12 marks 6. Consider the curve $y = f(x)$ given by

$$f(x) = (1 + e^x)^{-2},$$

answer the following questions. Show your work and justify your answers.

(a) Find all vertical asymptotes, if they exist.

(b) Find all horizontal asymptotes, if they exist.

(c) Find intervals of increase/decrease.

(d) Find local maxima and minima, if they exist.

(e) Find intervals where the functions are concave up/down.

(f) Find all inflection points, if they exist.

(g) Sketch the graph of f, indicating all features you identified above (e.g. indicate the coordinates of relevant points on the axes). Your sketch does need not be to scale.

6 marks

7. The gravitational attraction of the earth on a mass m at a distance r from the centre of the earth is a continuous function $F(r)$ given for $r \geq 0$ by

$$F(r) = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \geq R \\ mkr & \text{if } 0 \leq r < R. \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

(a) Find the constant k.

(b) Draw a rough sketch of the function $F(r)$. Make sure you identify the coordinates of relevant points.

(c) Show that, as a mass m moves away from the surface of the earth in an outward direction (i.e. away from the centre of the earth), the gravitational attraction F on the mass decreases at twice the rate at which F decreases if the mass were to move away from the surface of the earth in an inward direction (i.e. towards the centre of the earth).

$\boxed{\text{9 marks}}$ 8. Consider the first-order, autonomous differential equation

$$\frac{dy}{dt} = ky - cy^2,$$

where k and c are non-negative constants.

(a) Find the equilibria (constant solutions), and draw a rough graph of a few representative solutions $y = y(t)$. Make sure to identify the equilibrium solutions on your diagram.

(b) Now take $k = c = 1$, and use the Euler method with step size $\Delta t = 1/4$ to approximate the solution at $t = 1/2$ of the initial value problem $\frac{dy}{dt} = y - y^2, \quad y(0) = 2.$

(c) Now take $k = 0$, $c = 1$, and solve exactly the initial value problem $\frac{dy}{dt} = -y^2,\ y(0) = 2.$

$\boxed{\text{6 marks}}$ 9. Among all isosceles triangles of given perimeter, find the triangle with the greatest area. (Recall an isoscele triangle has two sides of equal length).

6 marks 10. Consider two functions f and g that are differentiable everywhere and such that the equation $f(x) = g(x)$ has two solutions.

(a) Prove that there exists (at least) one solution to the equation $f'(x) = g'(x)$.

(b) Show that there exist two numbers a and b so that the function

$$z(x) = \begin{cases} f(x) & \text{if } x \leq a \\ g(x) + b & \text{if } x > a \end{cases}$$

is continuous and differentiable everywhere.

$\boxed{\text{6 marks}}$ 11. (a) The Taylor polynomial of degree n for $\ln(x)$ centred at $x = 1$ is

$$T_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n.$$

Find the value of $\dfrac{d^{100}}{dx^{100}}(\ln(x))$ at $x = 1$.

(b) The Taylor polynomial of degree n for e^y centred at $y = 0$ is

$$T_n(y) = 1 + y + \frac{1}{2}y^2 + \frac{1}{3!}y^3 + \frac{1}{4!}y^4 + \cdots + \frac{1}{n!}y^n.$$

Find the value of $\dfrac{d^{12}}{dx^{12}}(e^{-x^4})$ at $x = 0$.

(c) Evaluate $\displaystyle\lim_{x\to 0}\left[\frac{\ln(1+x^4) + e^{-x^4} - 1}{x^{12}}\right]$.