ODEs

An **ordinary differential equation (ODE)** is an equation involving an unknown function $y(t)$ and its derivatives: $y'(t)$, $y''(t)$, etc. (and its variable, t).

- physics: $v(t) =$ velocity of object experiencing drag force proportional to the square of its speed: $\boxed{m\frac{dv}{dt} = -kv^2}$
- biology: $c(t) =$ concentration of DDT in a fish, accumulated at rate K, eliminated at rate proportional to c: $\boxed{c'(t) = K - kc(t)}$
- chem/phys: $\psi(x) =$ *wavefunction* of a free 1D quantum particle at fixed energy, solves Schrödinger's equation: $\boxed{\frac{d^2\psi}{dx^2} = -k^2\psi}$

Some terminology:

- **ordinary** : means the unknown function depends on *only one variable* (otherwise you have a **P**DE)
- an ODE is **first order** if only y and y' are involved (not y'', etc), **second order** if only y, y', y'' are involved, etc. ...

ODEs

A function satisfying an ODE is called a **solution** of it.
Given an ODE, our objective is usually either:

1. find all solutions of it (typically there are infinitely many)
   Example: find the general solution of $\frac{dy}{dt} = y$:
   $y(t) = Ce^t$ (how can we be sure these are *all* the solutions?)

   Example: find the **general solution** of $\frac{d^2y}{dx^2} = -y$:
   $y(x) = A\sin(x) + B\cos(x)$
2. find the solution satisfying some extra conditions, such as:
   - **initial conditions** (eg: initial position & velocity for Newton)
   - **boundary conditions** (eg: conditions on wavefunction at walls of box)

   Example: solve the **initial value problem** $\frac{dy}{dt} = y, \quad y(0) = 3$:
   $y(t) = Ce^t, \quad 3 = y(0) = C \quad \Longrightarrow \; y(t) = 3e^t$

ODEs

Sad fact: we can **not** solve most ODEs explicitly! So we must solve them **numerically** (approximately, by computer).
But there are a special few we can, such as:

First order, separable ODEs: $\frac{dy}{dt} = F(y)G(t)$.

Solve by separating variables, then integrating (antidifferentiating):

$$\frac{dy}{dt} = F(y)G(t) \quad \rightarrow \quad \frac{1}{F(y)}dy = G(t)dt \quad \rightarrow \quad \int \frac{1}{F(y)}dy = \int G(t)dt$$

Ex: find the general solution, then solve the initial value problem:

1. $\frac{dy}{dx} = \frac{e^y}{x^2}, \quad y(1) = 0$
2. (drag force) $\quad m\frac{dv}{dt} = -kv^2, \quad v(0) = v_0$
3. (DDT concentration) $\quad c'(t) = K - kc, \quad c(0) = 0$

ODEs

1. $\frac{dy}{dx} = \frac{e^y}{x^2}, \quad y(1) = 0$:
   gen. solution: $\int e^{-y}\,dy = \int x^{-2}\,dx \implies -e^{-y} = -x^{-1} + C$
   $\implies e^{-y} = \frac{1}{x} - C \implies y = -\ln(\frac{1}{x} - C)$
   initial value: $0 = y(1) = -\ln(1 - C) \implies C = 0$
   $\implies y(x) = -\ln(1/x) \implies \boxed{y(x) = \ln(x)}$.

2. (drag force) $m\frac{dv}{dt} = -kv^2, \quad v(0) = v_0$:
   $\int \frac{dv}{v^2} = -\int \frac{k}{m}dt \implies -\frac{1}{v} = -\frac{k}{m}t + C \implies v = \frac{1}{\frac{k}{m}t - C}$
   $v_0 = v(0) = -\frac{1}{C} \implies C = -\frac{1}{v_0} \implies \boxed{v(t) = \frac{v_0}{1 + \frac{kv_0}{m}t}}$.

3. (DDT in fish) $c'(t) = K - kc, \quad c(0) = 0$:
   $\int \frac{dc}{K - kc} = \int dt \implies -\frac{1}{k}\ln(K - kc) = t + C$
   $c(0) = 0 \implies C = -\frac{1}{k}\ln(K)$
   $\implies K - kc = e^{-kC}e^{-kt} = Ke^{-kt} \implies \boxed{c(t) = \frac{K}{k}(1 - e^{-kt})}$