# Science One Math

March 27, 2019

# Our goal: go back from numbers to functions!

**Our goal** today is to build an infinite series to represent a function

**Starting point**: a geometric series $\sum_{j=1}^{\infty} a\, r^{j-1} = \sum_{i=0}^{\infty} a\, r^i$

**Why a geometric series**? Because the geometric series converges to a well-known sum (for an appropriate choice of the "ratio" $r$)

$$\sum_{i=0}^{\infty} a\, r^i = \frac{a}{1-r} \quad \text{provided } |r| < 1$$

Examples of **geometric series**

$$3 + 3 \cdot \frac{2}{3} + 3 \cdot \frac{4}{9} + 3 \cdot \frac{8}{27} + \cdots = \sum_{n=0}^{\infty} 3\left(\frac{2}{3}\right)^n$$

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots = \sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n$$

$1 + a + a^2 + a^3 + \cdots = \sum_{n=0}^{\infty} a^n$ where $a$ is a constant

$1 - \frac{1}{2}(b-2) + \frac{1}{4}(b-2)^2 - \frac{1}{8}(b-2)^3 \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{(b-2)^n}{2^n}$
where $b$ is a constant

**$a$ and $b$ are PARAMETERS**

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$\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are PARAMETERS** ➥ If we treat the parameter as a *variable*,
we have a **power series**

If we treat the *parameter* as a *variable*, then we have a **power series**

$$1 + x + x^2 + x^3 + \cdots$$

or

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \frac{1}{8}(x-2)^3 + \cdots$$

think of these as “infinite polynomials”

# Terminology Power Series

A **power series** about $a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$a$ is fixed number, called ***centre*** of the series.
$\{c_n\}$ are the ***coefficients*** of the series.

This is also called **a power series in** $(\boldsymbol{x}-\boldsymbol{a})$.

*Example*: $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ power series centered at 0.

What is the centre of the series $\sum_{n=0}^{\infty} n^3 (2x - 3)^n$ ?

$A)\ a = 2/3$

$B)\ a = 3/2$

$C)\ a = 2$

$D)\ a = 3$

E) None of the above

# Do power series converge?

Power series can be used to define a function only if the series converges.

For what value(s) of $x$ does a power series converge?

➤ 3 possible cases: 1) convergence at a **point** (*centre*) *always*

2) convergence over an **interval** *sometimes*

3) convergence for **all** $\boldsymbol{x}$ *sometimes—ideal case*

E.g. $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ or $-1 < x < 1$ .

*Assumption*: When working with power series, we'll consider only $x$-values for which the series converges. We are not concerned with identifying the interval of convergence.

# What do power series converge to?

(one of) the most important series

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ for } |x| < 1.$$

If we treat the ratio $x$ as a variable, the sum of this series is a **function of $x$.**

# Can we build other functions from power series?

The function $\frac{1}{1-x}$ can be represented by $\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.

Can we build other functions from power series?

YES! ➢ by using manipulating the series $\sum_{n=0}^{\infty} x^n$

➢ by using $\sum_{n=0}^{\infty} c_n (x-a)^n$ identifying a general pattern for $c_n$

*(next week)*

# Manipulating power series

- Making a change of variable

*E.g.* We can express $\frac{1}{1+x}$ as a power series by making an appropriate substitution into $\sum_{n=0}^{\infty} x^n$.

# Manipulating power series

- Multiplying by a factor

*E.g.*

Express $\frac{3x}{2-x}$ as a power series.

*Strategy*: Multiplying a suitable geometric series by a factor containing $3x$

Which one of the following series can be used to represent $f(x) = \frac{4x^{12}}{1+3x}$ ?

*A.* $\sum_{n=0}^{\infty}(-1)^n 3^n x^n$

*B.* $\sum_{n=0}^{\infty} 4 \cdot 3^n x^n$

*C.* $\sum_{n=0}^{\infty} 4 \cdot 3^n x^{12+n}$

*D.* $\sum_{n=0}^{\infty}(-1)^n 4 \cdot 3^n x^{12+n}$

*E.* $\sum_{n=0}^{\infty}(-1)^n 4 \cdot 3x^{12+n}$

What is the sum of the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{9^{n+1}}$ ?

*A)* $\frac{x}{9+x^2}$

B) $\frac{x}{9-x^2}$

C) $\frac{x}{9+x}$

D) $\frac{x}{9-x}$

E) none of the above

Extra question: Express $\frac{1}{x}$ as a power series centred at 1.

Which of the following series converges to $\frac{1}{x}$ ?

$A)\ \sum_{n=0}^{\infty}(-1)^n(x-1)^n$

$B)\ \sum_{n=0}^{\infty}(x-1)^n$

$C)\ \sum_{n=0}^{\infty}(-1)^n(x)^n$

$D)\ -\sum_{n=0}^{\infty}(x-1)^n$

E) No idea

*Theorem*: Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on some interval I and let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then

- $Kf(x) = \sum_{n=0}^{\infty} Kc_n (x-a)^n$ where K is a constant
- $(x-a)^N f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n+N}$ for any integer $N \geq 1$
- $f$ is differentiable (hence continuous) on the interval of convergence
- $f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$
- $\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

(see Theorems 3.5.13 and 3.5.18 for list of operations on Power Series)

# Differentiating term by term

Express $\frac{1}{(1-x)^2}$ as a power series centred at 0 by differentiating $\sum_{n=0}^{\infty} x^n$ (for $|x| < 1$)

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) =$$

$$= 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

Note: summation of new series starts at $n = 1$.

# Integrating term by term

Problem: Express $\ln(1+x)$ as a power series centred at 0.

Find the series representation of $\arctan(x)$ .

# Most functions can be produced by manipulating $\sum x^n$ ...except...

... except important functions like $e^x$, $\sin(x)$, $\cos(x)$.

$\Rightarrow$ need to find a strategy to build appropriate $c_n$.

More generally,

- What is the power series representation of a function?
- Which functions have power series representations?

# Observation: the coefficients of a power series follow a pattern!

$f(x) = \frac{1}{1-x}$

$f'(x) = (1-x)^{-2}$ $\quad f''(x) = 2(1-x)^{-3}$ $\quad f'''(x) = 6(1-x)^{-4}$

$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 2, \quad f'''(0) = 6$

$$1 + x + x^2 + x^3 + \cdots = f(0) + f'(0) \cdot x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3 + \cdots$$

*same coefficients as in the Taylor polynomials!*

# (from Nov 7-8 slides) Taylor Polynomials

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note $T_n(x)$ is partial sum of $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

Recall: A series converges to $S$ if $\lim_{n\to\infty} S_n = S$, where $S_n$ are the partial sums.

**We want $\lim_{n\to\infty} T_n(x) = f(x)$.**

Recall error in approximating $f(x)$ with $T_n(x)$ is

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some $c$ between $a$ and $x$.

**If we let the degree $n$ to go to infinity, does the error go to zero?** *Yes, depends on f and x*

# **Taylor series:** a power series representation of a function

*Thrm*: If $f$ has a power series representation at $a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then the sequence generating the coefficients of the series is

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor series** of $f$ at $a$.

# Taylor series for $e^x$, $\sin(x)$ , $\cos(x)$

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

Question: Prove that $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges to $e^x$ for all $x$.