MODELLING WITH ODEs

Newton's Law of Cooling: the rate of change of an object's temperature is proportional to the difference between its temperature and that of its surroundings.

So temperature $T(t)$ satisfies $\boxed{\frac{dT}{dt} = -k[T - A]}$
where A is the temperature of the surroundings.

Can you solve this ODE?

- $\int \frac{dT}{T-A} = -k \int dt$
- $\ln |T - A| = -kt + C$
- $|T - A| = e^{-kt} e^{C}$
- at $t = 0$, $|T(0) - A| = e^{C}$
- so $|T - A| = e^{-kt} |T(0) - A|$
- $T - A = e^{-kt}(T(0) - A)$
- so $\boxed{T(t) = A + (T(0) - A)e^{-kt}}$

Crime scene math

The body of an apparent homicide victim is found in a room kept at $20^\circ C$. At 12 PM the temperature of the body is $35^\circ C$, and at 1 PM, it is $34.2^\circ C$. Find the time of death ,assuming the body is cooling according to Newton's law, and that its temperature at the time of death was $36.6^\circ C$.

- let $T(t)$ be body temperature (in $^\circ C$), t hours after noon
- $\frac{dT}{dt} = -k(T - 20) \implies T(t) = 20 + (35 - 20)e^{-kt}$
- $34.2 = T(1) = 20 + 15e^{-k}$
  $\implies e^{-k} = \frac{14.2}{15} \implies k = \ln\left(\frac{15}{14.2}\right)$
- find t_0 such that $36.6 = T(t_0) = 20 + 15e^{-kt_0}$
  $\implies e^{-kt_0} = \frac{16.6}{15} \implies t_0 = -\ln\left(\frac{16.6}{15}\right) / \ln\left(\frac{15}{14.2}\right)$
  ≈ -1.85, so time of death was $\approx 1h\, 51m$ before noon, i.e.
  $\approx 10:09\ AM$.

Population growth models (revisited)

Recall the **logistic equation**, a simple model for the growth of the population of some species in an environment with a finite *carrying capacity*:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

We have analyzed it *qualitatively*. Now we can solve it exactly:

- $\int \frac{K\,dP}{P(K-P)} = \int k\,dt = kt + C$
- $kt + C = \int \left(\frac{1}{P} + \frac{1}{K-P}\right) dP = \ln\left|\frac{P}{K-P}\right|$
- $C = \ln\left|\frac{P(0)}{K-P(0)}\right|$
- $e^{kt} = \left|\frac{P}{K-P}\,\frac{K-P(0)}{P(0)}\right| = \frac{P}{K-P}\,\frac{K-P(0)}{P(0)}$
- $\cdots$ algebra $\cdots$ $\boxed{P(t) = \frac{KP(0)}{P(0)+(K-P(0))e^{-kt}}}$

Does this expression accord with our previous qualitative analysis?

Object falling with drag (revisited)

An object, initially at rest, is released in a constant gravitational field and subject to drag forces proportional to its speed, and to the square of its speed. Write an ODE IVP to model this. Then contemplate how you might solve it.

$v(t) =$ velocity (downward) as a function of time:

$$m\frac{dv}{dt} = mg - av - bv^2, \quad v(0) = 0$$

$$\int \frac{m\,dv}{mg - av - bv^2} = \int dt = t + C$$

can be solved using partial fractions $\cdots$

A "mixing" problem

A river flows through a 10^7 m^3 lake at a rate of 10^4 m^3/day. A factory on the shore begins releasing chemical waste into the lake at a rate of 100 kg/day. Assume the lake is constantly well-mixed.

1. Write an ODE IVP for the mass of chemical waste in the lake as a function of time.
2. Solve it.
3. How much waste will be in the lake "in the long term"?

- let $M(t)$ be the mass (in kg) of waste in the lake after t days
- $\frac{dM}{dt}$ = rate in − rate out = $100 - \frac{M}{10^7}10^4$ (in kg/day), $M(0) = 0$
- $\int \frac{dM}{100-10^{-3}M} = \int dt \quad \Longrightarrow$
  $-10^3 \ln(100 - 10^{-3}M) = t + C = t - 10^3 \ln(100)$
- $100 - 10^{-3}M = 100e^{-\frac{t}{1000}} \quad \Longrightarrow \quad \boxed{M(t) = 10^5(1 - e^{-\frac{t}{1000}})}$
- $\lim_{t \to \infty} M(t) = 10^5$ kg.

Another "mixing" problem

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The well-mixed solution is drained from the tank at a rate of 3 L/min.

- Write an ODE for the amount of salt in the tank at time t.
- Can you solve it?

The difference here is that the volume of solution in the tank is not constant: the volume (in L) after t min. is $V(t) = 100 + 2t$.

So the mass (in kg) $M(t)$ of salt satisfies

$\frac{dM}{dt} = \text{rate in - rate out} = (0.4)5 - \frac{M}{100+2t}3$, so

$\frac{dM}{dt} = 2 - \frac{3}{100+2t}M, \quad M(0) = 0.$

This is *not* separable (but still easy to solve – you may learn how in a course next year...).

Reaction kinetics

Consecutive, first-order reactions: $A \xrightarrow{k_1} B \xrightarrow{k_2} C$

Write a **system** of ODEs for the concentrations $[A]$, $[B]$, $[C]$:

$$\frac{d[A]}{dt} = -k_1[A] \qquad \frac{d[B]}{dt} = k_1[A] - k_2[B] \qquad \frac{d[C]}{dt} = k_2[B]$$

How might you try to (start) solving this system?

- $[A](t) = e^{-k_1 t}[A]_0$
- $\frac{d[B]}{dt} = k_1[A]_0 e^{-k_1 t} - k_2[B]$ is not separable!

  But a simple trick (next year?) $\Longrightarrow$

  $[B](t) = [A]_0 \frac{k_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t}) + [B]_0 e^{-k_2 t}$
- plug this into ODE for $[C]$ and simply integrate ...

A geometrical problem

Find a family of curves, each of which intersects every parabola $y = Cx^2$ at right angles.

- slope of parabola through $(x, y = Cx^2)$ is $2Cx = 2\frac{y}{x^2}x = 2\frac{y}{x}$
- so at (x, y) our curve should have slope $-\frac{x}{2y}$:

$$\frac{dy}{dx} = -\frac{x}{2y} \quad \Longrightarrow \quad \int 2y\ dy = -\int x\ dx$$

$$\Longrightarrow \quad y^2 + \frac{1}{2}x^2 = C \qquad \text{ellipses!}$$