Reminders from our discussion of series last time...

The meaning of the **series** $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + a_4 + \cdots$:

Let $s_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$ be the n-th **partial sum**:

- if $\lim_{n\to\infty} s_n = s$, the series **converges**: $\sum_{j=1}^{\infty} a_j = s$.
- if $\lim_{n\to\infty} s_n$ does *not* exist, the series **diverges**.

A key example:

The **geometric series** $\sum_{j=1}^{\infty} ar^{j-1} = a+ar+ar^2+ar^3+ar^4+\cdots$

- converges, if $|r| < 1$, with $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$
- diverges, if $|r| \geq 1$

...by the usual limit laws applied to the sequences of partial sums:

If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent series, then so are $\sum_{j=1}^{\infty}(a_j \pm b_j)$ and $\sum_{j=1}^{\infty}(ca_j)$, with

- $\sum_{j=1}^{\infty}(a_j + b_j) = \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j$
- $\sum_{j=1}^{\infty}(a_j - b_j) = \sum_{j=1}^{\infty} a_j - \sum_{j=1}^{\infty} b_j$
- $\sum_{j=1}^{\infty}(ca_j) = c\sum_{j=1}^{\infty} a_j$

Ex: $\sum_{j=0}^{\infty} \frac{2^j - 1}{3^j} = \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j - \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j = \frac{1}{1-\frac{2}{3}} - \frac{1}{1-\frac{1}{3}} = 3 - \frac{3}{2} = \frac{3}{2}$

Does the series converge? if so, what is its value?

1. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

2. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

3. $\sum_{j=1}^{\infty} e^{-j}$

4. $\sum_{n=1}^{\infty} \arctan(n)$

5. $\sum_{j=1}^{\infty} \frac{1}{j(j+2)}$

6. $0.999999999\cdots$

Testing for convergence/divergence of series

Goal: check if a series converges or diverges *without computing any partial sums*.

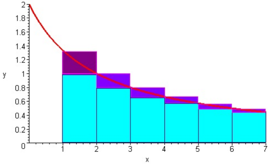
The simplest such **"convergence test"**, from last time:

Test for Divergence: if $\lim_{k\to\infty} a_k \neq 0$, $\sum_{k=1}^{\infty} a_k$ diverges.

Testing for convergence/divergence of series

Our next convergence test exploits the facts that:

- a series $\sum_{j=1}^{\infty} f(j)$ is similar to the improper integral $\int_1^{\infty} f(x)dx$



- integrals are easier to evaluate than sums

Integral Test: if $a_j = f(j)$ where f is positive, continuous, and decreasing on an interval $[N, \infty)$, then either

$$\sum_{j=1}^{\infty} a_j \quad \text{and} \quad \int_N^{\infty} f(t)dt \quad \text{both converge, or both diverge}$$

Warning: the integral test cannot tell you the **value** of a convergent series (only that it converges or diverges).

Convergence of p-series

$\sum_{k=1}^{\infty} \frac{1}{k^2}$: converges, since $\int_1^{\infty} \frac{dx}{x^2}$ converges (integral test)

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: diverges, since $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ diverges (integral test)

More generally, for which values of p does the

p-series $\sum_{j=1}^{\infty} \frac{1}{j^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ converge?

Since the integral $\int_1^{\infty} \frac{1}{x^p}\, dx$ converges if and only if $p > 1$:

the p-series converges if $p > 1$ and diverges if $p \leq 1$.

Example: the **harmonic series** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is the p-series with $p = 1$, so diverges.

Example: determine the convergence of:

- $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$: $\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{t\to\infty} \arctan(x)\big|_1^t = \frac{\pi}{2} - \frac{\pi}{4}$ converges,
  $\implies$ convergent, by integral test
- $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$: $\int_2^{\infty} \frac{1}{x\ln x} dx \; (u = \ln(x)) \; = \int_{\ln(2)}^{\infty} \frac{1}{u}\, du$ diverges,
  $\implies$ divergent, by integral test

Comparison Test

Just like with improper integrals, we can compare unfamiliar series with familiar ones to determine convergence:

> **Comparison Test**: suppose $0 \le a_n \le b_n$.
>
> - if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
> - if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

Examples: convergent or divergent?

1. $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{2^n+n}$ $\qquad$ $0 \le \frac{1+\sin(n)}{2^n+n} \le \frac{2}{2^n}$, $\sum_{n=1}^{\infty} \frac{2}{2^n}$ converges (geometric, $r = \frac{1}{2}$) $\implies$ <u>convergent</u>

2. $\sum_{k=2}^{\infty} \frac{k}{k^2-1}$ $\qquad$ $\frac{k}{k^2-1} \ge \frac{k}{k^2} = \frac{1}{k} \ge 0$, $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges (p-series, $p = 1$) $\implies$ <u>divergent</u>

3. $\sum_{j=2}^{\infty} \frac{j^2+3j+1}{j^4-j^3}$ $\qquad$ terms "behave like" $\frac{1}{j^2}$ for large j ..... ???

Limit Comparison Test

... a variant of the comparison test:

Limit Comparison Test: suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{for some } 0 < c < \infty.$$

Then either $\sum a_n$ and $\sum b_n$ both converge, or both diverge.

Example: $\sum_{j=2}^{\infty} \frac{j^2+3j+1}{j^4-j^3}$

compare with $\sum \frac{1}{j^2}$: $\lim_{j\to\infty} \frac{\frac{j^2+3j+1}{j^4-j^3}}{\frac{1}{j^2}} = \lim_{j\to\infty} \frac{j^4+3j^3+j^2}{j^4-j^3} = 1$,

so by the limit comparison test, since $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges

(p-series with $p = 2$), our series also converges.

Determine if each series converges or diverges:

1. $\sum_{k=1}^{\infty} \frac{3k+\sin(k)}{k^3 e^{-k}+k^2}$

2. $\sum_{n=1}^{\infty} \frac{(5n+7)^n}{(12n-1)^n}$

3. $\sum_{n=2}^{\infty} \frac{n+3}{n^2+n^2(\ln(n))^2}$

4. $\sum_{j=1}^{\infty} a_j, \qquad a_j = j\text{-th digit of } \pi$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

1. $\sum_{k=1}^{\infty} \frac{3k+\sin(k)}{k^3e^{-k}+k^2}$. Divergent by limit comparison with $\sum_{k=1}^{\infty} \frac{1}{k}$ (harmonic, so divergent): $\lim_{k\to\infty} \frac{\frac{3k+\sin(k)}{k^3e^{-k}+k^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{3+\frac{1}{k}\sin(k)}{ke^{-k}+1} = 3.$

2. $\sum_{n=1}^{\infty} \frac{(5n+7)^n}{(12n-1)^n}$. Convergent by limit comparison with $\sum_{n=1}^{\infty} \left(\frac{5}{12}\right)^n$ (geometric, $r = \frac{5}{12}$): let $c_n = \frac{\frac{(5n+7)^n}{(12n-1)^n}}{\left(\frac{5}{12}\right)^n} = \left(\frac{1+\frac{7}{5n}}{1-\frac{1}{12n}}\right)^n$ so that $\ln c_n = n\left(\ln(1+\frac{7}{5n}) - \ln(1-\frac{1}{12n})\right) = \frac{\ln(1+\frac{7}{5n})-\ln(1-\frac{1}{12n})}{\frac{1}{n}}$. By l'Hôpital, $\lim_{n\to\infty} \ln c_n = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\left(\frac{1}{1+\frac{7}{5n}}\frac{7}{5} - \frac{1}{1-\frac{1}{12n}}\left(-\frac{1}{12}\right)\right)}{-\frac{1}{n^2}} = \frac{89}{60}$ so $\lim_{n\to\infty} c_n = e^{\frac{89}{60}} \in (0,\infty)$.

3. $\sum_{n=2}^{\infty} \frac{n+3}{n^2+n^2(\ln(n))^2}$. Convergent since $\lim_{n\to\infty} \frac{\frac{n+3}{n^2+n^2(\ln(n))^2}}{\frac{1}{n(\ln(n))^2}} = 1$ and $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges by integral test: $\int_2^{\infty} \frac{dx}{x(\ln(x))^2} = \int_{\ln(2)}^{\infty} \frac{du}{u^2} < \infty$.

4. $\sum_{j=1}^{\infty} a_j, \qquad a_j = j$-th digit of π. $\boxed{\text{Divergent}}$ since $\lim_{j\to\infty} a_j \neq 0$.
(If the digits of π tended to 0, π would be rational.)

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. None of our tests so far work on this – stay tuned...

Each time a ball falls to the ground from height h, it rebounds to height rh, for some $0 < r < 1$. It is dropped initially from height H.

- Find the total distance the ball travels.
- Find the total time the ball travels for.