Solutions: Quiz Six, Nov 27th

Contains all solutions to all individual and group parts.

Problem: True/False: If matrices $A$ and $B$ are similar then they have the same eigenvalues and eigenvectors.

False: only the eigenvalues need to be the same (page 157 definition).

Problem: True/False: If matrices $A$ and $B$ have the same eigenvalues then they are similar.

False - they can have the same eigenvalues but fail to be similar. The statement is true if they are diagonalizable matrices.

Problem: True/False: If the sum of the eigenvalues of a matrix satisfies $\lambda_1 + \lambda_2 + \cdots + \lambda_n \neq 0$ then the matrix is always invertible.

False: consider \( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \). This has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$ so $\lambda_1 + \lambda_2 = 1$, but is clearly not invertible. However, if the product of all eigenvalues is nonzero then the matrix is invertible.

Problem: True/False: If the sum of the eigenvalues of a matrix is zero then the matrix is always invertible.

False: consider \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). This has eigenvalues $\{0, \pm 1\}$ which sum to zero. However, the determinant is obviously zero so it is not invertible.

Problem: True/False: It is possible to construct a real $2 \times 2$ matrix with eigenvalues 0 and $1+i$.

False: remember that complex eigenvalues of real matrices always come in complex conjugate pairs.

Problem: True/False: It is possible to construct a $3 \times 3$ matrix with eigenvalues $1$, $1-i$ and $1+i$.

True. Specifically, we could build as follows: \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \). (Check it!)

Problem: True/False: If $\lambda$ is an eigenvalue of a matrix $A$ then it is always an eigenvalue of $A^T$.

True. The characteristic polynomial does not change when you transpose the matrix.

Problem: Consider the non-zero $n \times n$ matrix $A = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}$. How many linearly independent eigenvectors does $A$ have?

Notice that $\text{rank}(A) = 1$ because the columns are all multiples of the first column. This also means that the null space of $A$ has dimension $n-1$. Therefore, $\lambda_1 = 0$ is an eigenvalue of $A$ with associated eigenspace of all the vectors that are in the null space. There must be $n-1$ linearly independent vectors in the null space since it has dimension $n-1$. We can say that $\lambda_1 = 0$ is an eigenvalue with multiplicity $n-1$.

Additionally, observe that $A\vec{x} = c(1,1,\ldots,1)^T$ for some scalar $c$ and any $\vec{x}$. This is the same as the eigenvalue equation $A\vec{x} = \lambda \vec{x}$ so $\vec{x} = (1,1,\ldots,1)^T$ is an eigenvector of $A$. A quick calculation shows that this $\vec{x}$ has eigenvalue $\lambda_2 = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. 

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Summarizing, we have found $n$ eigenvectors ($n-1$ linearly independent elements of the null space of $A$, and the other eigenvector $(1,1,\ldots,1)^T$). This makes $n$ eigenvectors. Finally, we know that $(1,1,\ldots,1)^T$ is linearly independent of the elements of the null space because it has a distinct associated eigenvalue (theorem 2 on page 150). So there are $n$ linearly independent eigenvalues.

**Problem:** For the matrix $A$ in the previous question, how many distinct eigenvalues does it have?

There are two distinct eigenvalues, 0 and 1.

**Problem:** Calculate the eigenvalues $\lambda_{1,2}$ of the matrix $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\det(B - \lambda I) = \lambda^2 - 2\lambda + 2$. Apply the quadratic formula to get $\lambda = 1 \pm i$. Alternatively, recognize that $B$ fits the discussion on pages 178-179 of the text and read off the eigenvalues.

**Problem:** Calculate the eigenvalues $\lambda_{1,2}$ of the matrix $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\det(B - \lambda I) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$. $\lambda_1 = 0, \lambda_2 = 2$.

**Problem:** Calculate the eigenvalues $\lambda_{1,2,3}$ of the matrix $B = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

This is an upper triangular matrix so the eigenvalues are just the diagonal entries (theorem 3 on page 155), or calculate explicitly: $\lambda_1 = -3, \lambda_2 = -2, \lambda_3 = -1$.

**Problem:** Suppose that $\vec{x}$ is an eigenvector of both the matrix $A$ and the matrix $B$, with corresponding eigenvalues $\lambda$ for $A$ and $\mu$ for $B$. Is $\vec{x}$ an eigenvector of $AB$, and if so, what is the corresponding eigenvalue $\phi$?

We need to try to solve the eigenvalue equation $AB\vec{x} = \phi\vec{x}$ for the eigenvalue $\phi$. We know that $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$.

$$AB\vec{x} = A\mu\vec{x} = \mu A\vec{x} = \mu\lambda\vec{x}.$$  
So $\phi = \lambda\mu$ is the eigenvalue of $AB$.

**Problem:** Suppose that $\vec{x}$ is an eigenvector of both the matrix $A$ and the matrix $B$, with corresponding eigenvalues $\lambda$ for $A$ and $\mu$ for $B$. Is $\vec{x}$ an eigenvector of $A+B$, and if so, what is the corresponding eigenvalue $\phi$?

We need to try to solve the eigenvalue equation $(A + B)\vec{x} = \phi\vec{x}$ for the eigenvalue $\phi$. We know that $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$.

$$(A + B)\vec{x} = A\vec{x} + B\vec{x} = \lambda\vec{x} + \mu\vec{x} = (\lambda + \mu)\vec{x}.$$  
So $\phi = \lambda + \mu$ is the eigenvalue of $A + B$.

**Problem:** Suppose that $\vec{x}$ is an eigenvector of both the matrix $A$ and the matrix $B$, with corresponding eigenvalues $\lambda$ for $A$ and $\mu$ for $B$. Suppose $A$ is invertible. Is $\vec{x}$ an eigenvector of $A^{-1}B$, and if so, what is the corresponding eigenvalue $\phi$?

We need to try to solve the eigenvalue equation $A^{-1}B\vec{x} = \phi\vec{x}$ for the eigenvalue $\phi$. We know that $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$.

$$A^{-1}B\vec{x} = A^{-1}\mu\vec{x} = \mu A^{-1}\vec{x} = \ldots.$$
Now what do we do? Observe that if \( A\vec{x} = \lambda \vec{x} \) then we can multiply both sides by \( A^{-1} \) and divide both sides by \( \lambda \) to find \( A^{-1} \vec{x} = (1/\lambda) \vec{x} \). Therefore, if \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( \vec{x} \), the \((1/\lambda)\) is an eigenvalue of \( A^{-1} \) with the same eigenvector \( \vec{x} \). So we can continue:

\[
A^{-1} B \vec{x} = A^{-1} \mu \vec{x} = \mu A^{-1} \vec{x} = \frac{1}{\lambda} \vec{x}.
\]

So \( \phi = \mu/\lambda \) is the eigenvalue of \( A^{-1} B \).

**Problem:** Suppose \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \{2 \pm i, 1\} \). What are the eigenvalues of \( A^{-1} \)?

See the previous solution and take the reciprocals of the given eigenvalues. In particular,

\[
\frac{1}{2 + i} = \frac{2 - i}{(2 + i)(2 - i)} = \frac{2 - i}{5} \quad \text{and} \quad \frac{1}{2 - i} = \frac{2 + i}{(2 - i)(2 + i)} = \frac{2 + i}{5}
\]

so we find the eigenvalues of \( A^{-1} \) are \( \{\frac{2}{5} \pm \frac{1}{5}i, 1\} \).

**Problem:** Suppose \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \{2 \pm i, 0\} \). What are the eigenvalues of \( A^{-1} \)?

Here, \( A^{-1} \) does not exist since there is a zero eigenvalue. Since there is a zero eigenvalue, that means there is a nonzero \( \vec{x} \) such that \( A \vec{x} = 0 \vec{x} = 0 \), and so the matrix has a nontrivial null space, or equivalently is not invertible.

**Problem:** Let \( A = \begin{pmatrix} -3 & 2 & a \\ 2 & -2 & b \\ 4 & 2 & 1 \end{pmatrix} \). Find \( a, b \) such that \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) is an eigenvector of \( A \).

This can be done by direct calculation, or by observing that \((1,1,1)^T\) can only be an eigenvector if the sum of each row of the matrix is the same. Either way, you should find that \( a = 8 \) and \( b = 7 \).

**Problem:** A population of grebes has two classes: juveniles and adults. Let the population in year \( n \) be given by the vector \( \vec{x}_n = \begin{pmatrix} j_n \\ a_n \end{pmatrix} \) where \( j_n \) and \( a_n \) indicate juveniles and adults at year \( n \), respectively.

Wildlife biologists determine that each year, one fifth of the juveniles mature to become adults in the following year, while five-sixths of the adults survive to the following year. Additionally, each adult has 5 offspring each year.

Can you find a grebe population vector \( \vec{x}_{\text{steady}} \) such that the proportions of juvenile and adult grebes remain the same from year to year? Mathematically, this would be written

\[
\lambda \vec{x}_{\text{steady}} = A \vec{x}_{\text{steady}}
\]

where \( A \) is the Leslie matrix (population update matrix). We see that \( \lambda \) and \( \vec{x}_{\text{steady}} \) are an eigenvalue-eigenvector pair for \( A \).

First we need to work out the Leslie matrix from the description of the model: \( A = \begin{pmatrix} 0 & 5 \\ 1/5 & 5/6 \end{pmatrix} \).

To find the steady populations, you need to find the eigenvalues and eigenvectors of this matrix. These are \( \lambda_1 = 3/2 \) and \( \lambda_2 = -2/3 \), with corresponding eigenvectors \( \vec{x}_1 = \begin{pmatrix} 10 \\ 3 \end{pmatrix} \) and \( \vec{x}_2 = \begin{pmatrix} -15 \\ 2 \end{pmatrix} \). Since \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \), we will find that the long-term behaviour of the population will be governed by \( \lambda_1 \) and \( \vec{x}_1 \). Thus the steady population is \( \vec{x}_{\text{steady}} = \vec{x}_1 = \begin{pmatrix} 10 \\ 3 \end{pmatrix} \) and this population will grow by a factor of \( \lambda_1 = 3/2 \) every year.
Problem: There are two main baseball teams in New York. The teams are called the Yankees and the Mets. Baseball fans in New York always support one team or the other, but never both. Surveys revealed that in year 2000, 75% of the fans supported the Yankees while 25% supported the Mets. However, we also know that every year, 10% of the Mets fans switch to supporting the Yankees. Also, every year, 5% of the Yankees fans switch to supporting the Mets. Let’s encode the population preferences in year $n$ as the vector $\vec{x}_n = \begin{pmatrix} Y_n \\ M_n \end{pmatrix}$ where $Y_n$ and $M_n$ are the proportions of the Yankees and the Mets fans respectively.

What proportion of the Yankees fans would we need to have in order that the proportion does not change from year to year? (We call this an equilibrium or steady state of the system.)

We begin by setting up this problem using a matrix:

$$
\begin{pmatrix}
Y_{n+1} \\
M_{n+1}
\end{pmatrix} =
\begin{pmatrix}
.95 & .10 \\
.05 & .90
\end{pmatrix}
\begin{pmatrix}
Y_n \\
M_n
\end{pmatrix}.
$$

Now calculate the eigenvectors and eigenvalues: $\lambda_1 = 1$ and $\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\lambda_2 = 17/20$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since $A\vec{x}_1 = 1\vec{x}_1$ this means that $\vec{x}_1$ is the steady state proportion, so at steady state there will be 2 Yankees fans for every 1 Mets fan.