Quiz 4 (103/104) Individual

1. \textbf{TRUE} ; follows by invertible matrix theorem

\begin{align*}
A \text{ invertible} &\iff \text{columns of } A \text{ form a basis of } \mathbb{R}^n \\
A^T \text{ invertible} &\iff \text{columns of } A^T \text{ form a basis of } \mathbb{R}^n
\end{align*}

\[ \text{columns of } A^T = \text{rows of } A \]

2. \textbf{FALSE};

Take \( \mathbb{R}^2 \): its subspaces are \( \{0\}^2 \), \( \mathbb{R}^2 \) and lines through the origin.

Union of two lines through the origin is neither \( \{0\} \), nor \( \mathbb{R}^2 \) nor a line through origin.
Alternatively: the union of two lines through origin is not closed under addition.

\[ u \in H_1 \cup H_2 \]
\[ v \in H_1 \cup H_2 \]
\[ u + v \notin H_1 \cup H_2 \]

(3) \[ \text{answer: } \]
\[ H_3 \text{ only} \]

\[ H_1: \text{ does not contain the origin} \]
\[ (0,0,0,0) \notin H_1 \text{ b/c } 3 \cdot 0 + 10 \cdot 0 - 6 \cdot 0 + 2 \cdot 0 \neq 1 \]
so not a subspace

\[ H_2: \text{ not closed under addition} \]
\[ (1,-1,1,1) \in H_2 \text{ b/c } 1 \cdot 1 + (1) \cdot 1 = 0 \]
\[ (2,1,3,-6) \in H_2 \text{ b/c } 2 \cdot 3 + 1 \cdot (-6) = 0 \]

but their sum \[ (3,0,4,-5) \notin H_2 \text{ b/c } 3 \cdot 4 + 0 \cdot (-5) \neq 0 \]
so not a subspace
\[ H_3 : \]
\[
\begin{align*}
3x_1 + 10x_2 - 6x_3 + 2x_4 &= 0 \\
x_2 - 5x_3 + x_4 &= 0
\end{align*}
\]

can be viewed as

\[
\text{Nul} \left( \begin{bmatrix} 3 & 10 & -6 & 2 \\ 0 & 1 & -5 & 1 \end{bmatrix} \right)
\]

so indeed a subspace.

\[ \boxed{\text{Answer: } \begin{pmatrix} -8 \\ 5 \end{pmatrix}} \]

matrix of linear map \( T^{-1} \) is

\[
[T]^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1}
\]

\[
= \frac{1}{1 \cdot 3 - 2 \cdot 2} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}
\]

\[
= \frac{1}{-1} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}
\]

matrix of \( x \mapsto S(T^{-1}(x)) \) is

\[
[S] \cdot [T]^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -8 & 5 \\ 10 & -5 \end{pmatrix}
\]
5. **Answer:** \(16 \cdot \text{det}(A)\)

For a 4x4 matrix

\[
\text{det}(-2A) = (-2)^4 \text{det}(A)
\]

one factor of \(-2\) per each row

**General Rule:** A \(n \times n\) matrix, \(c\) a scalar

\[
\text{det}(cA) = c^n \text{det}(A)
\]

6. **Answer:** 1015

**The Rank Theorem**

\[
\dim \text{Nul}(A) + \text{rank}(A) = \frac{2015}{1000}
\]

so \(\text{rank}(A) = 2015 - 1000 = 1015\)
(7) \[ \text{answer: (1), (2) and (3) (all equivalent to } A \text{ being invertible)} \]

(1) and (2) are equivalent to \( A \) being invertible
(by the invertible matrix theorem)

(3) says there exists an invertible matrix \( B \) such that \( BA \) is invertible (through lenses of the IMT)

If \( B \) is invertible and \( BA \) is invertible then
\[ B^{-1} (BA) = (B^{-1}B)A = A \]
is invertible as well.
\[ \begin{vmatrix} + & - & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 2015 & 0 & 2020 \\ 3 & 2016 & 0 & 2019 & 0 \\ 0 & 2017 & 0 & 2020 & 4 \\ 0 & 2018 & 5 & 2021 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} + & - & + & - \\ 0 & 0 & 2 & 0 \\ 3 & 0 & 2019 & 0 \\ 0 & 0 & 2020 & 4 \\ 0 & 5 & 2021 & 0 \end{vmatrix} \]

\[ = (-1) \cdot 2 \begin{vmatrix} + & - \\ 3 & 0 & 0 & 4 \\ 0 & 5 & 0 \end{vmatrix} \]

\[ = (1) \cdot 2 \cdot 3 \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} \]

\[ = (-1) \cdot 2 \cdot 3 \cdot (-20) = 120 \]