1) **Answer:** TRUE

This follows from The Invertible Matrix Theorem:

\[ A \text{ is invertible } \iff A^T \text{ is invertible} \]

\[ \iff \text{rank}(A^T) = n \]

\[ (a \text{ way of saying that } \text{RREF}(A^T) = I_n) \]

2) **Answer:** TRUE

This follows from compatibility of definitions of a subspace and a linear transformation.

Want to show: if \( y_1, y_2 \in T(H) \) and \( c_1 \) and \( c_2 \) are scalars, then \( c_1 y_1 + c_2 y_2 \in T(H) \). (i.e. \( T(H) \) is closed under taking linear combinations.)
\[ y_1 = T(x_1) \quad \text{for some } x_1 \in \mathbb{R}^n \]
\[ y_2 = T(x_2) \quad \text{for some } x_2 \in \mathbb{R}^n \]

Then
\[ c_1 y_1 + c_2 y_2 = c_1 T(x_1) + c_2 T(x_2) \]
\[ = T(c_1 x_1 + c_2 x_2) \in T(H) \]

an element of \( \mathbb{R}^n \)

So we produced an element in \( \mathbb{R}^n \) whose image is precisely \( c_1 y_1 + c_2 y_2 \) which makes \( c_1 y_1 + c_2 y_2 \in T(H) \)

as desired.

(3) \( \boxed{\text{answer: } H_3 \text{ only}} \)

\( H_1 : \) does not contain the origin \( b/c \)
\[
(0,0,0,0) \notin H_1 \quad b/c \quad 0 - 7.0 + 11.0 - 5.0 \neq 1
\]
so not a subspace
$H_2$: not closed under addition

\[(1, 1, 1, 0) \in H_2 \text{ b/c } 1 \cdot 1 \cdot 0 = 0\]
\[(1, 1, 0, 1) \in H_2 \text{ b/c } 1 \cdot 0 \cdot 1 = 0\]

but their sum \[(2, 2, 1, 1) \notin H_2 \text{ b/c } 2 \cdot 1 \cdot 1 \neq 0\]

so not a subspace

$H_3$:

\[x_1 - 7x_2 + 11x_3 - 5x_4 = 0\]
\[x_2 - 3x_4 = 0\]

$H_3$ can be viewed as

\[\text{Nul } \left[ \begin{array}{cccc} 1 & -7 & 11 & -5 \\ 0 & 1 & 0 & -3 \end{array} \right]\]

so indeed a subspace.
matrix of lin. trans. $S^{-1}$

$$[S]^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1}$$

$$= \frac{1}{1.5 - 2.3} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ (-1) \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

matrix of $x \mapsto S^{-1}(T(x))$

$$[S]^{-1} [T] = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & -3 \end{pmatrix}$$
$$\text{answer: } 16 \cdot \det(A)$$

for a $4 \times 4$ matrix

$$\det(2A) = 2^4 \cdot \det(A)$$

one factor of 2 per each row

**General Rule:** A $n \times n$ matrix, $c$ a scalar

$$\det(cA) = c^n \det(A)$$

$$\text{answer: } 1016$$

The Rank Theorem $\Rightarrow$

$$\dim \text{ Nul}(A) + \text{rank}(A) = \frac{2016}{1000} \quad \# \text{ of columns}$$

so $\dim \text{ Nul}(A) = 2016 - 1000 = 1016$
answer: (1), (2) and (3)
(all equivalent to $A$ being invertible)

(1) $\iff A$ invertible directly by
The Invertible Matrix Theorem

(2) $\iff A^T$ invertible $\iff A$ invertible

(3) says that there exists an invertible matrix $B$ such that
$BA$ is invertible

Then $B^{-1}(BA) = A$ must be invertible as well.
8. \[ \text{answer: } -1 \]

This is a "triangular" matrix so keeping track of signs the value is \[ (1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot 1 = -1 \]

Alternatively, follow the same strategy as for usual upper/lower triangular matrices (with respect to the main diagonal).