Consider a $2 \times 2$ matrix $A$. When a vector $v$ satisfies

$$v \neq 0,$$

$$Av = \lambda v$$

then we say that $v$ is an eigenvector of $A$ of eigenvalue $\lambda$. We note

$$A(kv) = k(Av) = k(\lambda v) = \lambda(kv),$$

which says that non zero multiples of eigenvectors yield more eigenvectors of the same eigenvalue. Let us first consider the geometric transformations we previously mentioned. An eigenvector will correspond to a direction that is fixed (or reversed) by the transformation.

$$D(2,3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

will have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 2 and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigenvector of eigenvalue 3. The identity matrix $I$ has the property that any non zero vector $v$ is an eigenvector of eigenvalue 1.

The rotation matrix $R(\theta)$ has no eigenvectors, by the geometric reasoning that no directions are preserved, unless $\theta = 0, \pi$. There will be no (real) roots of the quadratic.

The shear matrix $G_{12}(\gamma) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 1 but no other eigenvectors (other than multiples) for $\gamma \neq 0$.

The following analysis is critical in seeking eigenvectors and eigenvalues:

there exists a $v$ with $Av = \lambda v; \quad v \neq 0$

if and only if there exists a $v$ with $Av = \lambda I v; \quad v \neq 0$

if and only if there exists a $v$ with $(A - \lambda I)v = 0; \quad v \neq 0$

if and only if $\det(A - \lambda I) = 0$

Now

$$\det(A - \lambda I) = \det \left( \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \right) = \lambda^2 - (a + d)\lambda + (ad - bc)$$

$$= \lambda^2 - \text{tr}(A)\lambda + \det(A),$$

which (for $2 \times 2$ matrices) is a quadratic function in $\lambda$ and whose roots you can seek by standard methods.

Sample computation

$$A = \det \left( \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \right)$$

$$\det(A - \lambda I) = \det \left( \begin{bmatrix} .7 - \lambda & .3 \\ 2 & -\lambda \end{bmatrix} \right)$$
\[(.7 - \lambda)(-\lambda) - .3 \times 2\]
\[= \frac{1}{10}(10\lambda^2 - 7\lambda - 6)\]
\[= \frac{1}{10}(5\lambda - 6)(2\lambda + 1)\]

Thus we have two eigenvalues \(\lambda = \frac{6}{5}, -\frac{1}{2}\).

For \(\lambda = \frac{6}{5}\), we solve \((A - \frac{6}{5}I)v = 0\) for \(v \neq 0\):
\[(A - \frac{6}{5}I)v = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

The vector \(v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}\) works as an eigenvalue of \(A\) of eigenvalue \(\frac{6}{5}\). We check
\[\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5} \begin{bmatrix} 3 \\ 5 \end{bmatrix} .\]

For \(\lambda = -\frac{1}{2}\), we solve \((A - \frac{1}{2}I)v = 0\) for \(v \neq 0\):
\[(A - \frac{1}{2}I)v = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

The vector \(v = \begin{bmatrix} 1 \\ -4 \end{bmatrix}\) works as an eigenvalue of \(A\) of eigenvalue \(-\frac{1}{2}\). We check
\[\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -4 \end{bmatrix} .\]

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue \(\lambda\) has \(\det(A - \lambda I) = 0\).

The origin of this matrix was a model of bird populations. Let
\[x_n = \text{no. of adults in year } n,\]
\[y_n = \text{no. of juveniles in year } n.\]

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized as
\[x_{n+1} = .7x_n + .3y_n\]
\[y_{n+1} = 2x_n\]

which can be written as a matrix equation:
\[\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.\]

Note that this operation would work in many settings of abstracting a matrix equation from a system of equations. We deduce by induction, that
\[\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} .\]