1. Solve the following differential equations for $y(t)$:

   a. $\frac{dy}{dt} = 6t^2 - \frac{y}{t}$, with $y(1) = 1$.

   \[ y^1 + \frac{1}{t}y = 6t^2 \]
   \[ \mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t \]
   \[ (t \frac{dy}{dt}) = 6t^3 \]
   \[ ty = \frac{3}{2}t^4 + C \]
   \[ y(1) = 1 \Rightarrow 1 = \frac{3}{2} + C \]
   \[ C = -\frac{1}{2} \]
   \[ y = \frac{3}{2}t^3 - \frac{1}{2}t^{-1} \quad \text{Solution:} \quad y = \frac{3}{2}t^3 - \frac{1}{2}t^{-1} \]

   b. $\frac{dy}{dt} = (y - y^2)te^{t^2}$, with $y(0) = 2$.

   \[ \int \frac{dy}{y-y^2} = \int te^{t^2} \, dt \]
   \[ \int \frac{dy}{y(y-1)} = \int \frac{A}{y} - \frac{A}{y-1} \, dy = \frac{1}{y-1} \]
   \[ A(1-y) - Ay = 1 \]
   \[ A = 1 = B \]
   \[ \int \frac{dy}{y} + \int \frac{dy}{y-1} = \ln |y| - \ln |y-1| \]
   \[ \ln \left| \frac{y}{y-1} \right| = \frac{1}{2}e^{t^2} + C \]

   \[ y(0) = 2 \]
   \[ \ln 2 = \frac{1}{2} + C \]
   \[ C = \ln 2 - \frac{1}{2} \]
   \[ y = \frac{\frac{1}{2}e^{t^2} + 1}{1 - 2e^{t^2} - \frac{1}{2}} \quad \text{Solution:} \quad y = \frac{\frac{1}{2}e^{t^2} + 1}{1 - 2e^{t^2} - \frac{1}{2}} \]
2. Match the direction fields with the differential equations by circling A, B, C, or D in each case. Some of these differential equations do not match any of the direction fields - for those cases circle “none”.

(1) \( y' = y^2(2 - y) \)  
    A \( \bigcirc \) B C D none

(2) \( y' = y(2 - y) \)  
    A B C D \( \text{none} \)

(3) \( y' = y(2 - y)^2 \)  
    A \( \bigcirc \) B C D none

(4) \( y' = y - 2x \)  
    A B C \( \bigcirc \) D none

(5) \( y' = y - \sin x \)  
    A B C D \( \text{none} \)

(6) \( y' = y + 2x \)  
    A B \( \bigcirc \) C D none
3. Consider the following system of differential equations with initial conditions.

\[ x'(t) = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} x(t) \quad x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \]

a. Solve the system for \( x_1(0) = 1, x_2(0) = 0 \).

**Eigenvalues:**
\[ \lambda^2 - 5\lambda + 0 = 0 \]
\[ \lambda(\lambda - 5) = 0 \]
\[ \lambda = 0, 5. \]

**Eigenvectors:**
\[ \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} \]
\[ \lambda = 5 \]
\[ \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

**Solution:**
\[ x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} \]

**Initial Conditions:**
\[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]
\[ c_2 - 2c_1 = 0 \Rightarrow c_2 = 2c_1 \]
\[ c_1 + 2c_2 = 0 \Rightarrow 5c_1 = 0 \]
\[ c_1 = 1/5, c_2 = 2/5. \]

\[ x = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} \]
3 b. Find all possible initial conditions for the system so that $|x(t)|$ does not go to infinity as $t \to \infty$.

Let $c_2 = 0$ so that it avoids blow up.

\[ x(t) = c_1 \left( \frac{1}{2} \right) + O \left( \frac{t^2}{t^2} \right) e^{st}, \quad c_1 \text{ arbitrary} \]

Hence $x(0) = c_1 \left( \frac{1}{2} \right)$

And $x(t) = c_1 \left( \frac{1}{2} \right)$
4. For this question, assume that turkeys follow Newton’s law of cooling/heating. I start my oven heating at 12pm. The temperature of the oven increases linearly with time until it reaches 150°C at 12.30pm. The oven is initially at room temperature, 20°C.

a. Write an equation for the temperature of the oven as a function of \( t \), the time in hours since 12pm.

\[
T = \frac{150 - 20}{112} = 260^\circ \text{C}
\]

\[
T_0 - 20 = 260(t - 0)
\]

\[
T = 20 + 260t.
\]

b. A turkey is placed in the oven at 12pm. The relaxation time of the turkey is 1 hour. (Hint: this means that the constant in Newton’s law is \( \pm 1/\text{hour} \) depending on your sign convention). The temperature of the turkey at 12pm is 4°C. Find the temperature of the turkey at 12.30pm (leave powers of \( e \) as part of your answer).

\[
\frac{dT}{dt} = -1(T - T_\text{ov})
\]

\[
\frac{dT}{dt} + T = 20 + 260t
\]

\[
T = e^{\int dt} = e^t + C = e^t
\]

\[
e^t(T) = e^t(20 + 260t)
\]

\[
e^tT = 20e^t + \int_{20}^{20 + 260t} e^u \, du
\]

\[
= 20e^t + 260te^t - \int e^t \, du
\]

\[
e^tT = 120e^t + 260te^t - 260e^t + C
\]

\[
T = 260t - 240 + Ce^{-t}
\]

\[
T(\frac{1}{2}) = 120 - 240 + Ce^{-1/2} = -110 + 244e^{-1/2} \degree \text{C}
\]

\[
T(0) = 4 \Rightarrow 4 = -240 + C
\]

\[
C = 244
\]