Problem 1: Determine the Laplace transform $F(s) = \int_{0}^{\infty} e^{-st} f(t) dt$ of the following by evaluating the integral:

1. $f_1(t) = te^{3t}$
2. $f_2(t) = e^{-t} \sin(2t) + \cos(3t)$

Solution:

1. $Y_1(s) = \frac{1}{(s-3)^2}$
2. $Y_2(s) = \frac{s}{s^2+9} + \frac{2}{(s+1)^2+4}$

Problem 2: Use a Laplace transform table (e.g. Boyce & Diprima Table 6.2.1 or see Wikipedia), the linearity of the Laplace transform, and trigonometric identities to determine the Laplace transforms of these functions:

1. $f_3(t) = 2t^2 e^{-t} - t + \cos(4t)$
2. $f_4(t) = e^{7t} \sin^2(t)$
3. $f_5(t) = \cos(nt) \sin(mt), \ m \neq n$

Solution:

1. $Y_3(s) = \frac{4}{(s+1)^2} - \frac{1}{s} + \frac{s}{s^2+16}$
2. $Y_4(s) = \frac{-s-7}{2(s-7)(s^2+4)} + \frac{1}{2(s-7)}$ (use double-angle formula to replace $\sin^2(t)$ with a cosine function).
3. $Y_5(s) = \frac{1}{2} \left( \frac{m+n}{(m+n)^2+s^2} - \frac{n-m}{(m-n)^2+s^2} \right)$ [Write $\cos(nt) \sin(mt) = (1/2)(\sin((n+m)t) - \sin((n-m)t))$.]

Problem 3: Use the definition of Laplace transform to calculate $F(s)$ in the following case:

$f(t)$ for Problem 1
Solution:
\[ F(s) = \int_0^\infty f(t)e^{-st} \, dt \]
\[ = \int_0^1 te^{-st} \, dt + \int_1^2 (t-2)e^{-st} \, dt \]
\[ = \frac{1 - 2se^{-s} - e^{-2s}}{s^2} \]

Problem 4: Use a table of Laplace transforms to find \( y(t) \) in each case.

Use partial fractions, frequency shift, time shift, or other rules as needed.

1. \( Y(s) = \frac{s}{(s+1)^2+1} \)

2. \( Y(s) = \frac{s^2}{(s+2)^2(s+4)} \)

3. \( Y(s) = \frac{3s+1}{(s-4)^2+2} \)

4. \( Y(s) = \frac{1+e^{-2t}}{s-4} \)

5. \( Y(s) = \frac{1}{(s-3)^2} \)

6. \( Y(s) = \frac{s^2-2}{s^3} \)

Solution:

1. \( Y(s) = \mathcal{L}[\cos(t)](s + 1) - \frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\cos(t)](s) - \mathcal{L}[e^{-t}\sin(t)](s) \implies y(t) = e^{-t}(\cos(t) - \sin(t)) \)

2. \( Y(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+4} \) for \( A = -3, B = 2 \) and \( C = 4 \).

Note that \( \frac{1}{(s+2)^2} = \mathcal{L} \left[ \frac{1}{\pi t} \right](s+2) = \mathcal{L}[e^{-2t}](s) \)

Together: \( y(t) = -3e^{-2t} + 2e^{-2t}t + 4e^{-4t} \)

3. Rewrite entirely in terms of \( (s - 4) \):

\[ Y(s) = \frac{3(s - 4) + 13}{(s - 4)^2 + 2} = \frac{3}{(s - 4)} + \frac{13}{\sqrt{2}} + \frac{\sqrt{2}}{(s - 4)^2 + 2} \]

\[ = 3\mathcal{L} \left[ \cos(\sqrt{2}t) \right] (s - 4) + \frac{13}{\sqrt{2}} \mathcal{L} \left[ \sin(\sqrt{2}t) \right] (s - 4) \]

\[ = 3\mathcal{L} \left[ e^{+4t}\cos(\sqrt{2}t) \right](s) + \frac{13}{\sqrt{2}} \mathcal{L} \left[ e^{+4t}\sin(\sqrt{2}t) \right](s) \]

Conclude \( y(t) = e^{+4t} \left( 3\cos(\sqrt{2}t) + \frac{13\sqrt{2}}{2}\sin(\sqrt{2}t) \right) \)

4. Separate factors of different exponentials:

\[ Y(s) = \frac{1}{s^2 - 4} + e^{-2s} \frac{1}{s^3 - 4} \]

Then, by partial fractions,

\[ \frac{1}{s^2 - 4} = \frac{1}{4} \left( \frac{1}{s - 2} - \frac{1}{s + 2} \right) \implies \frac{1}{s^2 - 4} = \mathcal{L} \left[ \frac{e^{2t}}{4} - \frac{e^{-2t}}{4} \right](s) \]
For the other term, use the time-shift rule,

\[ e^{-2s}F(s) = \mathcal{L}[u_2(t)f(t-2)] \]

Together:

\[ y(t) = \left( \frac{e^{2t}}{4} - \frac{e^{-2t}}{4} \right) + u_2(t) \left( \frac{e^{2(t-2)}}{4} - \frac{e^{-2(t-2)}}{4} \right) \]

This can be simplified, but that is not required.

5.

\[ Y(s) = \frac{1}{(s-3)^3} = \frac{1}{3!} \mathcal{L}[t^3] (s-3) = \frac{1}{6} \mathcal{L}[e^{3t}t^3] (s) \rightarrow y(t) = \frac{1}{6} e^{3t}t^3 \]

6.

\[ Y(s) = \frac{s^2 - 2}{s^3} = 1 - \frac{2}{s^3} = \mathcal{L}[1](s) - \mathcal{L}[t^2](s) \rightarrow y(t) = 1 - t^2 \]

**Problem 5 or 6:** Use Euler’s method with \( h = 0.1 \) to estimate \( y(1) \) for the following two problems, and compare with the exact answer. Next, use the Improved Euler method with \( h = 0.1 \) to solve the problems again. Is your answer better or worse? Finally compare with the solution obtained via `ode45` in Matlab/Octave.

Note: you can use Matlab to implement Euler and Improved Euler, or you can use excel (scripts and spreadsheet available with this homework).

**Solution:** Solution will be updated soon for this problem.