Show all relevant work for credit. You will be marked for your work and your answer as appropriate. Talking to other students about the problems is encouraged but you must submit your own work and identify who you worked with at the top of your assignment.

ALL WORK MUST BE STAPLED. WRITE YOUR NAME AND STUDENT NUMBER AT THE TOP OF THE FIRST PAGE. MESSY WORK WILL NOT BE GRADED.

Problem 1: In each case, solve for \( y(t) \):

a. \( y' + 2y = 0 \) with \( y(0) = 1 \).

b. \( y' + 2y = -3e^t \) with \( y(0) = -2 \).

c. \( ty' - 2y = t^3 \sin t \) with \( y(\pi) = 0 \).

Solution: Problem 1:

a. Use integrating factor \( e^{2t} \) to get \( ye^{2t} = C \). With initial condition we get,
\[
y(t) = e^{-2t}
\]

b. Use integrating factor \( e^{2t} \) to get \( ye^{2t} = -e^{3t} + C \). With initial condition we get,
\[
y(t) = -e^t - e^{-2t}
\]

c. Divide by \( t \), then use integrating factor \( e^{-2ln t} = \frac{1}{t^2} \) and \( \frac{d}{dt} \left( \frac{y}{t^2} \right) = \sin t \). With initial condition we get,
\[
y(t) = -t^2(\cos(t) + 1)
\]

Problem 2: Use Matlab (dirfield) to generate a direction field for each equation from problem 1. Make plots on the range \( 0 < t < 4, \quad -4 < y < 4 \). Superimpose plots (ezplot) of the analytic solutions corresponding to your answers from problem 1.

Submit: the commands you used to achieve this, and your properly labelled plots.

See the last pages of this assignment for notes on plotting functions and generating direction fields in Matlab. To combine multiple plots together you can use the commands hold on and hold off. After obtaining the first plot type hold on, then all subsequent commands plot in the same window. After the last plot command type hold off.

Solution: Problem 2:

a. \( y' + 2y = 0 \) with \( y(0) = 1 \).
b. \( y' + 2y = -3e^t \) with \( y(0) = -2 \).
c. \( ty' - 2y = t^3 \sin t \) with \( y(\pi) = 0 \).

```matlab
% Define the function f=y'
f = @(t,y) (t^3*sin(t)+2*y)/t;

% Call dirfield command with appropriate t and y discretization
dirfield(f,0:.5:4,-4:1:4);

% Use hold on to superimpose the analytic solution
hold on

% Plot the analytical solution on the same interval
ezplot('-t^2*(cos(t)+1)',[0,4,-4,4])
hold off

% Adjust labels
xlabel('t');
ylabel('y');
title('Prob. 2B − Direction field & solution curve');
```
Problem 3: (a) Use Matlab to generate the slope field of \( y' = y - \sin(t) + 1 \) on \( 0 < t < 2, \, -4 < y < 4 \). Obtain the solution curve for initial condition \( y(0) = -1 \) using \texttt{ode45} and superimpose on the slope field.

Submit: the commands you typed into Matlab to achieve this, and your properly labelled plot.

(b) Suppose that the initial condition is \( y(0) = \alpha \) and \( \lim_{t \to \infty} y(t) = -\infty \). What can you say about \( \alpha \)? To resolve this question you can make a guess based on the direction field plot, but you will need to solve the equation analytically to be confident of your answer.

Solution: Problem 3:

\begin{verbatim}
% Define the function f(t,y)=y'
f = @(t,y) y-sin(t)+1;

% Call dirfield command with appropriate t and y discretization
dirfield(f,0:.25:2,-4:0.5:4);

% Use hold on to superimpose the analytic solution
hold on

% Obtain the numerical solution using ode45, saving as (ts,ys)
[ts,ys] = ode45(f,[0 2],[-1]);
\end{verbatim}
(b) By integrating factor, the general solution is:

\[ y(t) = \frac{\cos(t) + \sin(t) - 2}{2} + \text{Const} \ e^t \]

Note that \( \lim_{t \to \infty} y(t) = -\infty \) only if \( \text{Const} < 0 \). For \( y(0) = \alpha \), we get \( \text{Const} = \alpha + \frac{1}{2} \).

If \( \lim_{t \to \infty} y(t) = -\infty \), then \( \alpha < -\frac{1}{2} \).

Problem 4: Your lab partner leaves a drop of bleach on the lab bench, which takes the shape of a hemisphere. The drop initially has a radius of 1.6mm, and evaporates at a rate proportional to its surface area. After 10 minutes, the radius is 1.5mm. How long until the drop is gone?

Hint: you can solve this by generating a differential equation for the volume of the drop, \( \frac{dV}{dt} = \ldots \) or with a little more work, a differential equation for the radius of the drop, \( \frac{dr}{dt} = \ldots \). The latter equation is much easier to solve.
Solution: Problem 4:

We are asked to find $r(t)$ where,

$$\frac{d}{dt} \left( \frac{1}{2} \frac{4}{3} \pi (r(t))^3 \right) = K \left( \frac{1}{2} 4\pi r(t)^2 \right),$$

for some unknown (negative) constant $K$. You can apply the chain rule to get

$$\frac{dr}{dt} = K \Rightarrow r(t) = K t + C$$

Use $r(0) = 1.6$ and $r(10) = 1.5$ to find that the drop will disappear after 160 minutes.

Problem 5: Find all values of $r$ so that the function $y = x^r$ solves the differential equation

$$x^2 y'' - 3xy' - 32y = 0.$$

Solution: Problem 5:

We let $y = x^r$ so that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Then $x^2 y'' = r(r-1)x^r$ and $-3xy' = -3rx^r$, so $x^2 y'' - 3xy' - 32y = [r(r-1) - 3r - 32]x^r = 0$.

The solutions to $r(r-1) - 3r - 32 = r^2 - 4r - 32 = 0$ are $r = 8$ and $r = -4$.

Problem 6: In a murder investigation a corpse is found at exactly 8:00 p.m. The temperature of the body is immediately measured and is found to be 70°F (Fahrenheit). Two hours later, it is again measured and found to be 60°F. If the room temperature is 50°F, and we assume that Newton’s law of cooling applies, when did the murder occur? (Assume that the temperature of the body at the time of the murder was 98.6°F).

Newton’s law of cooling states that the temperature $T$ of an object in a room at temperature $T_0$ follows the equation

$$\frac{dT}{dt} = k(T_0 - T).$$

Solution: Problem 6: Let $T(t)$ be the temperature of the body in degrees Fahrenheit starting at $t = 0$, which measures hours starting from 8:00p.m. Then, $T(t)$ satisfies

$$T' = -k(T - 50), \quad T(0) = 70.$$  \hspace{1cm} (4.1)

We also know that $T(2) = 60$. The solution to (4.1) is

$$T = 50 + 20e^{-kt}. \hspace{1cm} (4.2)$$

Now set $T = 60$, when $t = 2$ to determine $k$ as

$$k = \ln(2)/2 \approx .34657...$$

Now we want to find $t$ such that $T = 98.6$. Clearly, this value of $t$ will satisfy $t < 0$, as the murder occurs before 8:00p.m. Thus,

$$98.6 = 50 + 20e^{-kt}, \quad t = -k^{-1} \ln(48.6/20) = -2.562 \approx -2.6.$$
Thus, the murder occurs approximately 156 minutes before 8:00p.m. or at 5:24p.m.

**Problem 7:** In the previous problem, you probably concluded that the time of the murder was about 2.6 hours before the first temperature reading. However, someone points out your analysis is faulty because the room temperature in which the corpse was found was not constant but instead decreased exponentially according to the law $50e^{-0.05t}$, where $t$ is the time (in hours), where $t = 0$ corresponds to 8:00 p.m.

- (i) What is the differential equation that you must solve now?
- (ii) Solve the equation to find an analytic expression for the temperature of the body at any time $t$.
- (iii) Use Matlab (for example `ezplot`, see the end of this document) to make an accurate plot of the body temperature as a function of time. Estimate the time of the murder. Submit: your plot and the estimated time of death, as well as the commands you used.

**Solution: Problem 7:**

- (i) The differential equation is now
  \[ T' = -k(T - 50e^{-0.05t}), \quad T(0) = 70. \]

- (ii) Multiply by $e^{kt}$ and integrate to get
  \[ T(t) = \frac{50k}{(k - 0.05)} e^{-t/20} + ce^{-kt}. \]

Now satisfy $T(0) = 70$, which determines $c$. Thus,
\[ T(t) = \frac{50k}{(k - 0.05)} e^{-t/20} + \left(70 - \frac{50k}{(k - 0.05)}\right) e^{-kt}. \]

- (iii) This last part is quite difficult and requires knowledge of Newton’s method and access to a programmable calculator. Note that it is difficult to determine $k$ by setting $T(2) = 60$, since we have a nonlinear equation for $k$ to solve. This equation can be written in the form
  \[ 60 - 50e^{-1} = \frac{3}{k} + 20e^{-2k} - \frac{3.5}{k}e^{-2k}. \]

Using Newton’s method to approximate the solution to this equation gives $k \approx .29$. Now the murder occurs at a time $\tau$ when $T = 98.6$. This Newton’s method is used to solve for $\tau$, which gives $\tau = -3.5$. Thus, the murder occurred 210 minutes before 8:00p.m. or at 4:30p.m.

**Problem 8:** Stephan’s law of radiative cooling states that the temperature $T$ of a material at time $t$ satisfies
\[ T' = -k\left(T^4 - T_{env}^4\right) \quad (*) \]

where $k > 0$ is constant and $T_{env}$ is the constant environmental temperature. This law holds for bodies heated to a very high temperature that will emit blackbody radiation (i.e. molten iron).
• (i) By separating variables and factoring \( T^4 - T_{env}^4 = (T^2 - T_{env}^2)(T^2 + T_{env}^2) \), show that the general solution to this equation is

\[
\ln \left( \frac{T + T_{env}}{T - T_{env}} \right) + 2 \tan^{-1} \left( \frac{T}{T_{env}} \right) = 4T_{env}^3 k t + c
\]

where \( c \) is a constant.

• (ii) Let \( T(0) = T_0 > T_{env} \). Calculate \( \lim_{t \to \infty} T(t) \).

Solution: Problem 8:

• (i) Separate variables and factor \( (T^4 - T_{env}^4) = (T^2 - T_{env}^2)(T^2 + T_{env}^2) \). Then, use partial fractions to get

\[
\frac{1}{(T^4 - T_{env}^4)} \, dT = -k dt \quad \to \quad \frac{1}{2T_{env}^2} \left( \frac{1}{T - T_{env}} - \frac{1}{T + T_{env}} \right) \, dT = -k dt . \tag{11.1}
\]

Now use partial fractions on the first term to get

\[
\frac{1}{2T_{env}^2} \left[ \frac{1}{2T_{env}} \left( \frac{1}{T - T_{env}} - \frac{1}{T + T_{env}} \right) - \frac{1}{(T^2 + T_{env}^2)} \right] \, dT = -k dt .
\]

Finally, we integrate each of the terms to get

\[
\ln \left( \frac{T + T_{env}}{T - T_{env}} \right) + 2 \tan^{-1} \left( \frac{T}{T_{env}} \right) = 4T_{env}^3 k t + c . \tag{11.2}
\]

The constant \( c \) can be chosen to satisfy the initial condition.

• (ii) As \( t \to \infty \), the right side in (11.2) tends to \( \infty \). Therefore, there must be a term on the left side of (11.2) that also tends to \( \infty \). Since \( |2 \tan^{-1} (x)| < \pi \) for any real number \( x \) and \( \ln y \to \infty \) as \( y \to \infty \), we conclude that \( T \to T_{env}^+ \) as \( t \to \infty \), for then \( -\ln(T - T_{env}) \to \infty \) as \( T \to T_{env}^+ \).