Question 1

(5 points) Solve the following initial value problems for \( y(t) \): \( ty' + 2y = \cos(t), \ y(\pi) = 0 \).

\[
y' + \frac{2}{t}y = \frac{1}{t}\cos(t)
\]

integrating factor is

\[
h(t) = e^{\int \frac{2}{t} \, dt} = t^2
\]

\[
\frac{d}{dt}(t^2y) = t\cos(t)
\]

integrating,

\[
t^2y = t\sin(t) + \cos(t) + c_1
\]

\[
y(t) = \frac{1}{t}\sin(t) + \frac{1}{t^2}\cos(t) + \frac{c}{t^2},
\]

\[
y(\pi) = 0 \implies c = 1
\]

\[
y(t) = \frac{1}{t}\sin(t) + \frac{1}{t^2}\cos(t) + \frac{1}{t^2}.
\]
(10 points) Consider the following system of first order ODEs

\[
\frac{dy_1}{dt} = 2y_1(t) - 2y_2(t) \\
\frac{dy_2}{dt} = 2y_1(t) + 2y_2(t)
\]

(a) Find the general solution of the system. Convert any complex exponentials in your solutions into "real forms" involving sines and cosines.

Let \( A = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \)

For the eigenvalues, \( \lambda^2 - 4\lambda + 8 = 0 \) and this gives \( \lambda_1 = 2 + 2i \) and \( \lambda_2 = 2 - 2i \).

For the eigenvector \( \vec{v}_1 \),

\[
\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}
\]

For the eigenvector \( \vec{v}_2 \),

\[
\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}
\]

We know that the general solution of the system is given by

\[
\vec{y}(t) = \text{Re} \left( \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t} \right) + \text{Im} \left( \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t} \right)
\]

Consider

\[
\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t} = e^{2t} \begin{pmatrix} i \\ 1 \end{pmatrix} \left( \cos(2t) + i \sin(2t) \right) = e^{2t} \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + i e^{2t} \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}
\]

Therefore, the general solution is

\[
\vec{y}(t) = c_1 e^{2t} \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}
\]

(b) Use the initial conditions \( y_1(0) = 1 \) and \( y_2(0) = 2 \) to find the constants in your solution.

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = 2, \ c_2 = 1.
\]

Therefore,

\[
\vec{y}(t) = 2 e^{2t} \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + e^{2t} \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}
\]
Question 3

(7 points) Determine the value of $k$ for which the following equation is exact

$$(y \cos(x) + kxe^y) \, dx + (\sin(x) + x^2e^y - 1) \, dy = 0.$$ 

For this equation to be exact, we need $M_y = N_x$.

$$M(x, y) = y \cos(x) + kxe^y \implies M_y(x, y) = \cos(x) + kxe^y$$

$$N(x, y) = \sin(x) + x^2e^y - 1 \implies N_x(x, y) = \cos(x) + 2xe^y$$

Comparing $M_y$ and $N_x$, we have that $k = 2$.

Therefore, the equation is

$$(y \cos(x) + 2xe^y) \, dx + (\sin(x) + x^2e^y - 1) \, dy = 0,$$

Let $\Phi(x, y) = C$ be the general solution of this equation. Then

$$\frac{\partial \Phi}{\partial x} = M(x, y) = y \cos(x) + 2xe^y \implies \Phi(x, y) = y \sin(x) + x^2e^y + \gamma_1(y)$$

$$\frac{\partial \Phi}{\partial y} = N(x, y) = \sin(x) + x^2e^y - 1 \implies \Phi(x, y) = y \sin(x) + x^2e^y - y + \gamma_2(x)$$

Comparing the two $\Phi(x, y)$ functions, we have that $\gamma_1(y) = -y$ and $\gamma_2(x) = 0$.

Therefore, the general solution is

$$y \sin(x) + x^2e^y - y = C$$

Applying the initial condition $y(\pi) = 0$, we have $C = \pi^2$.

$$y \sin(x) + x^2e^y - y = \pi^2.$$ 

Question 4

(i) $C$

(ii) \( \vec{y}(t) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) as $t \longrightarrow \infty$

(iii) B
Question 5

(12 points) Consider the following ODE
\[ \frac{dy}{dt} = \lambda (y^2 - 4), \quad \text{where } \lambda > 0. \]

(a) Find all the equilibria (steady state solutions) of the differential equation. For steady state solutions, we set \( \frac{dy}{dt} = 0 \) and solve for \( y \) in
\[ \lambda (y^2 - 4) = 0, \]
which gives \( y = -2 \) and \( y = 2 \).

(b) Sketch the graph of \( \frac{dy}{dt} vs y(t) \) and use it to determine which of these equilibria is stable, unstable, or semi-stable.

(c) Use the initial condition \( y(0) = 1 \) to find a solution to the equation. Hint: You may need the partial fraction
\[ \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}. \]
Thus,
\[ \frac{dy}{dt} = \lambda (y^2 - 4), \]
\[ \frac{1}{(y-2)(y+2)} = \lambda \frac{A}{y-2} + \frac{B}{y+2} \quad \Rightarrow \quad A = \frac{1}{4}, \quad B = \frac{-1}{4} \]
Therefore, we have
\[ \int \left( \frac{1}{4 (y-2)} - \frac{1}{4 (y+2)} \right) dP = rt + c_1, \]
\[ \frac{1}{4} \ln(y-2) - \frac{1}{4} \ln(y+2) = \lambda t + c_1, \]
\[ \frac{(y-2)}{(y+2)} = c_2 e^{4\lambda t}, \quad (c_2 = e^{4c_1}) \]
\[ y(t) = \frac{2 + 2c_2 e^{4\lambda t}}{1 - c_2 e^{4\lambda t}} \]
Applying the initial condition \( y(0) = 1 \), we have \( c_2 = -\frac{1}{3} \).
\[ y(t) = \frac{2 - \frac{2}{3} e^{4\lambda t}}{1 + \frac{2}{3} e^{4\lambda t}} = \frac{6 - 2e^{4\lambda t}}{3 + e^{4\lambda t}} \]

(d) Find the limit of this solution \( y(t) \) as \( t \to \infty \).
\[ \lim_{t \to \infty} y = \lim_{t \to \infty} y(t) = \frac{6 - 2e^{4\lambda t}}{3 + e^{4\lambda t}} = -2. \]