Question 1

(7 points) Solve the following initial value problems for $y(t)$:

$$ty' + (1 - 2t)y = 2te^{3t}, \quad y(1) = 2.$$ 

\begin{align*}
y' + \left(\frac{1}{t} - 2\right)y &= 2e^{3t} \\
\text{integrating factor is} \quad h(t) &= e^{\int \left(\frac{1}{t} - 2\right) dt} = e^{-2t} \\
\frac{d}{dt}(te^{-2t}y) &= 2te^t \\
\text{integrating,} \quad te^{-2t}y &= 2te^t - 2e^t + c \\
y(t) &= 2e^{3t} - \frac{2}{t}e^{3t} + \frac{c}{t}e^{2t}, \\
y(1) &= 2 \implies c = 2e^{-2} \\
y(t) &= 2e^{3t} - \frac{2}{t}e^{3t} + \frac{2}{t}e^{2(t-1)}.
\end{align*}
(7 points) Consider the following system of first order ODEs

\[
\begin{align*}
\frac{dy_1}{dt} &= y_1(t) + 3y_2(t) \\
\frac{dy_2}{dt} &= 3y_1(t) + y_2(t)
\end{align*}
\]

(a) Find the general solution of the system.

Let \( A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \)

For the eigenvalues, \( \lambda^2 - 2\lambda - 8 = 0 \) and this gives \( \lambda_1 = 4 \) and \( \lambda_2 = -2 \).

For the eigenvector \( \vec{v}_1 \),

\[
\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

For the eigenvector \( \vec{v}_2 \),

\[
\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

Therefore the general solution of the system is

\[
\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}.
\]

(b) Use the initial conditions \( y_1(0) = 0 \) and \( y_2(0) = -2 \) to find the constants in your solution.

\[
\begin{pmatrix} 0 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies c_1 = -1, \ c_2 = -1.
\]

Therefore,

\[
\vec{y}(t) = -1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}.
\]
Question 3

(8 points) Consider the following initial value problem:

\[(xy^3) \, dx + (x^2y^2 + 1) \, dy = 0, \quad y(1) = 1\]

(a) Is the differential equation exact?

\[M(x, y) = xy^3 \quad \Rightarrow \quad M_y(x, y) = 3xy^2\]
\[N(x, y) = x^2y^2 + 1 \quad \Rightarrow \quad N_x(x, y) = 2xy^2\]

Since \(M_y \neq N_x\), the equation is **NOT EXACT**.

b) If YES, find a solution that satisfies the given initial condition. Otherwise, find an integrating factor \(h(y)\) and use it to find a solution to the equation that satisfies the given initial condition.

Let us find an integrating factor \(h(y)\) of the equation

\[\frac{N_x - M_y}{M} = \frac{2xy^2 - 3xy^2}{xy^3} = -\frac{1}{y},\]
\[\frac{dh}{dy} = \left(-\frac{1}{y}\right) h(y)\]

Integrating, we have

\[h(y) = \frac{1}{y}\]

Multiplying the ODE by \(h(y) = \frac{1}{y}\), we have

\[(xy^2) \, dx + \left(x^2y + \frac{1}{y}\right) \, dy = 0,\]

The general solution of the equation

\[\frac{x^2y^2}{2} + \ln(y) = c\]

Applying the initial condition \(y(1) = 1\), gives \(c = \frac{1}{2}\). Therefore, the solution is

\[\frac{x^2y^2}{2} + \ln(y) = \frac{1}{2}\]

**Question 4**

(i) B

(ii) \(y(t) \to 0\) as \(t \to \infty\)

(iii) D
(12 points) The population of a certain bacteria grows according to the following differential equation

\[
\frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P, \quad \text{where } r, K > 0
\]

(a) Explain the meaning of the parameters \( r \) and \( K \).

- \( r \) is intrinsic growth rate
- \( K \) is the carrying capacity of the environment in which the bacteria is growing

(b) Find all the equilibria (steady state solutions) of the differential equation. For steady state solutions, we set \( \frac{dP}{dt} = 0 \) and solve for \( P \) in

\[
r \left(1 - \frac{P}{K}\right) P = 0,
\]

which gives \( P = 0 \) and \( P = K \).

(c) Sketch the graph of \( \frac{dP}{dt} \) vs \( P \) and use it to determine which of these equilibria is stable, unstable, or semi-stable.

(d) Use the initial condition \( P(0) = P_0 \) \((P_0 > 0)\) to find the population \( P(t) \) of the bacteria. Hint: You may need the partial fraction \( \frac{1}{P(K-P)} = \frac{A}{P} + \frac{B}{(K-P)} \).

\[
\frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P, \quad \text{where } r, K > 0
\]

\[
\Rightarrow \quad \frac{K \, dP}{(K-P) \, P} = r \, dt
\]

but

\[
\frac{1}{P(K-P)} = \frac{A}{P} + \frac{B}{(K-P)} \quad \Rightarrow \quad A = B = \frac{1}{K}
\]

Therefore, we have

\[
\int \left( \frac{1}{P} + \frac{1}{K-P} \right) dP = rt + c_1,
\]

\[
\ln(P) - \ln(K-P) = rt + c_1,
\]

\[
\frac{P}{(K-P)} = c_2 e^{rt}
\]

\[
P = \frac{c_2 Ke^{rt}}{1 + c_2 e^{rt}}
\]

Applying the initial condition \( P(0) = P_0 \) gives

\[
P = \frac{P_0 Ke^{rt}}{K + (e^{rt} - 1)P_0}
\]
OR

\[ P = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}} \]

(e) What is the population \( P(t) \) as \( t \rightarrow \infty \).

\[ \lim_{t \to \infty} P = \lim_{t \to \infty} \frac{P_0K}{P_0 + (K - P_0)e^{-rt}} = K. \]