Welcome to Math 215/255
Ordinary Differential Equations

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• Office hours:
  – Tues 10.30-11.30, Weds 11-12 and by appointment
• Course web site:
  • http://www.math.ubc.ca/~coombs/215/math215255.html

Clickers

• Clickers will be used in class regularly from Monday, Sept 11 onward. Bring your clicker to class.
• Please make sure your clicker is registered through connect.

This week’s topics:

• Today: What is a differential equation (DE)
• Classification of DEs: Linear vs nonlinear, 1st order, 2nd order, etc
• Method of separation of variables and Integrating Factor method for 1st order
• (next week) Slope fields
• Reading: Lebl 0.2, 1.1-1.4.
Differential Equations

are equations that link functions to their own derivatives.

e.g. solve \( \frac{df}{dt} = f(t) \) for \( f(t) \)

a solution would be \( f(t) = e^t \)
\( f(t) = 3 e^t \)
\( f(t) = Ce^t \) for some constant \( C \)

or even
\( f(t) = 0 \)

Ordinary differential equations (ODEs) have only one independent variable
so we solve for \( f(t) \)

Partial differential equations (PDEs) have multiple independent variables
so we solve for example \( f(x, y, z, t) \)

example: solve \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) for \( u(x, t) \).
The order of an ODE is the highest number of derivatives in the problem

\[
\frac{d^4 y}{dt^4} = \sin(y) + 2 \quad \text{is 4th order}
\]

\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2 = t^2 \quad \text{is 2nd order}
\]

Mostly, we will look at 1st and 2nd order in this class.

Linear vs. Nonlinear

If an ODE contains only linear functions of \( y(t) \), \( y'(t) \), \( y''(t) \), etc then it is linear.

1st order ODE are linear only if they can be written in the form

\[
a(t) \frac{dy}{dt} + b(t) y = c(t)
\]

2nd order version:

\[
a(t) \frac{d^2 y}{dt^2} + b(t) \frac{dy}{dt} + c(t) y = d(t)
\]
All other equations are **nonlinear**.

**Linear ODE:**  
⇒ we have solutions everywhere we might reasonably expect.  
⇒ we have a decent toolkit for finding the solutions  
⇒ numerically and analytically we are OK.

**Nonlinear ODE**  
⇒ sometimes no solution or multiple solutions  
⇒ no general techniques  
⇒ in practice work numerically (which can be very hard) or approximately (which can be very hard).
Separation of Variables

If a first-order ODE can be written in the form

\[ M(t) + N(y) \frac{dy}{dt} = 0 \]

then it is “separable” and solvable by integration.

\[ \int M(t) \, dt = - \int N(y) \frac{dy}{dt} \, dt \]

Do the integrals and solve for \( y(t) \).

Example

Solve \( \frac{dy}{dx} = \frac{x^2}{y} \), \( y(0) = 1 \)

\[ \int y \frac{dy}{dx} \, dx = \int x^2 \, dx \]

\[ \frac{y^2}{2} = \frac{x^3}{3} + C \]

\[ y = \pm \sqrt{\frac{2x^3}{3} + C} \]

Use \( y(0)=1 \) to find out \( C \).
choose \( + \sqrt{0 + C} \) \( \quad \) \( C = +1 \)

\[
\text{Solve:} \quad y(x) = +\sqrt{\frac{2x^3}{3} + 1}
\]

Observe: we had to choose \( +/- \)
and if \( x \) is too negative, this solution fails to exist.

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**Example 1st order linear**

\[
\frac{dy}{dt} + \frac{2}{t} y = 3
\]

**Magic:** multiply through by \( t^2 \).

\[
t^2 \frac{dy}{dt} + 2t y = 3t^2
\]

\[
\frac{d}{dt} (t^2 y) = 3t^2
\]

\[
t^2 y = \int 3t^2 \, dt
\]

\[
t^2 y = t^3 + C
\]

\[
y = t + \frac{C}{t^2}
\]

(reverse product rule)

(integrate both sides)

Check: plug \( y \) into the original equation.
Integrating Factor Method
(solves any 1st order linear ODE)

\[
a(t) \frac{dy}{dt} + b(t) y = c(t) \div a(t)
\]

\[
\frac{dy}{dt} + p(t) y = g(t)
\]

\[
p(t) = \frac{b(t)}{a(t)}
\]

\[
g(t) = \frac{c(t)}{a(t)}
\]

Goal: make LHS into \( \frac{d}{dt} (\text{something}) \)

Multiply through by arbitrary \( \mu(t) \)

\[
\mu(t) \frac{dy}{dt} + \mu(t) p(t) y(t) = g(t) \mu(t)
\]

\[
\frac{d}{dt} \left[ \mu(t) y(t) \right] = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt} y(t)
\]

Need: \( \frac{d\mu}{dt} = \mu(t) p(t) \)

\[
\int \frac{1}{\mu(t)} \, d\mu = \int p(t) \, dt
\]

\[
\ln \mu = \int p(t) \, dt + C
\]

\[
\mu = Ce^{\int p(t) \, dt}
\]

choose \( C = 1 \) here.
\[
\frac{d}{dt} \left[ e^{\int p(t) dt} y(t) \right] = g(t) e^{\int p(t) dt}
\]

int: \quad e^{\int p(t) dt} y(t) = \int g(t) e^{\int p(t) dt} dt + C

solve for y:
\[
y(t) = e^{-\int p(t) dt} \int g(t) e^{\int p(t) dt} dt + C e^{-\int p(t) dt}
\]

This formula solves every linear 1st order ODE.