Welcome to Math 215/255
Ordinary Differential Equations

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• Office hours:
  – Tues 10.30-11.30, Weds 11-12 and by appointment
• Course web site:
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Clickers

• Clickers will be used in class regularly from Monday, Sept 11 onward. Bring your clicker to class.
• Please make sure your clicker is registered through connect.

This week’s topics:

• Today: What is a differential equation (DE)
• Classification of DEs: Linear vs nonlinear, 1st order, 2nd order, etc
• Method of separation of variables and Integrating Factor method for 1st order
• (next week) Slope fields
• Reading: Lebl 0.2, 1.1-1.4.
Differential Equations are equations that link functions with their derivatives:
\[ \frac{df}{dt} = f \] solve for \( f(t) \)
\[ f(t) = Ae^t \] where \( A \) is any constant.

Ordinary Differential Equations have only one independent variable.

\[ \text{ex: Solve for } f(t) \]

Partial Differential Equations have multiple independent variables.

\[ \text{ex: solve for } f(x, y, z) \]

so a P.D.E. could be \[ \frac{df}{dt} = \frac{\partial^2 f}{\partial x^2} \]
solve for \( f(x, t) \).

Classification of O.D.E.

Order of an ODE is the highest number of derivatives that appears.

\[ y'' + y = 0 \] is 2nd order

\[ y'''' + y = 0 \] is 4th order.
$y'' + y' + y = 6$ is 2\textsuperscript{nd} order

Note: the number of constants in the solution is equal to the order.

This class: mostly 1\textsuperscript{st} and 2\textsuperscript{nd} order.

\underline{Linear vs Nonlinear}

\textbf{Defn} Any ODE that is not linear is nonlinear.

First order linear can be written as:

$$a(t) \frac{dy}{dt} + b(t) y = c(t) \quad \text{(for } y(t))$$

Linear equations cannot include any functions of $y(t)$, or $\frac{dy}{dt}$ etc. Functions of $t$ are OK.

Second order linear can be written as:

$$a(t) \frac{d^2y}{dt^2} + b(t) \frac{dy}{dt} + c(t) y = d(t)$$
Linear

⇒ we will have solutions where they reasonably can be expected to exist.

⇒ we have a reasonable toolkit for finding the solutions (I can always write the solution) in terms of some integrals.

⇒ for a lot of interesting or practical situations, we can compute the solution by hand.

Nonlinear

⇒ no guarantee a solution even exists (or is unique).

⇒ generically we are not ok.

⇒ in practical work, go numerically or approximately.
Separation of Variables for 1st Order ODE

If a 1st order ODE can be written in the form

\[ M(t) + N(ty) \frac{dy}{dt} = 0 \]

then we call the equation "separable" and solve by integration:

\[ \int M(t) \, dt = -\int N(ty) \frac{dy}{dt} \, dt \]

Do the integrals and solve for \( y(t) \)

Ex: Solve \( \frac{dy}{dx} = \frac{x^2}{y} \), \( y(0) = 1 \)

\[ y \frac{dy}{dx} = x^2 \]

\[ \int y \frac{dy}{dx} \, dx = \int x^2 \, dx \]

\[ \frac{y^2}{2} = \frac{x^3}{3} + C \]

\[ y = \pm \sqrt{\frac{2x^3}{3} + C} \]
Apply $y(0) = 1$ to solve for $C$:

$$1 = \sqrt{0 + C}$$

$$y = \sqrt{\frac{2c^3}{3}} + 1$$

Choose $C = 1$ here.

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Integrating Factor Method

solves any 1st order linear ODE

$$a(t) \frac{dy}{dt} + b(t) y(t) = c(t)$$

$$\frac{dy}{dt} + p(t) y(t) = g(t)$$

Let $$p = \frac{b}{a}, \quad g = \frac{c}{a}$$

Now multiply through by an arbitrary function $\mu(t)$. Our goal will be to choose $\mu(t)$ so that the LHS of the equation is equal to $\frac{d}{dt} [\mu(t) y(t)]$

Multiply:

$$\mu(t) \frac{dy}{dt} + \mu(t) p(t) y(t) = \mu(t) g(t)$$

$$\left[ \mu(t) y(t) \right] = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt} y(t)$$
Need to choose \( \mu \) so that \( \frac{d\mu}{dt} = \mu(t) p(t) \).

\[
\int \frac{1}{\mu} \frac{d\mu}{dt} = \int p(t) \, dt
\]

\[
\ln \mu = \int p(t) \, dt
\]

\[
\mu = e^{\int p(t) \, dt}
\]

(no need for constant here only)

My equation is:

\[
\frac{1}{dt} \left[ e^{\int p(t) \, dt} y(t) \right] = e^{\int p(t) \, dt} g(t)
\]

Integrate both sides:

\[
e^{\int p(t) \, dt} y(t) = \int e^{\int p(t) \, dt} g(t) \, dt + C
\]

\[
y(t) = e^{-\int p(t) \, dt} \int e^{\int p(t) \, dt} g(t) \, dt + C e^{-\int p(t) \, dt}
\]