Euler's Method

\[
\frac{dy}{dt} = f(t, y) \quad y(0) = y_0
\]

Suppose we want to know \(y(T) \quad T > 0\).

\[
y_1 = y_0 + h \left. \frac{dy}{dt} \right|_{t=0} = y_0 + hf(0, y_0)
\]

\[
y_2 = y_1 + hf(h, y_1)
\]

\[
y_3 = y_2 + hf(2h, y_2)
\]

\[
y_{k+1} = y_k + hf(t_k, y_k) \quad (t_k = kh)
\]

after \(N\) steps choose \(h = \frac{T}{N}\).
The obvious problem with Euler's method is that we are only calculating the derivative at certain points, and it may change in between those points. Also, the error in the derivative may grow at every step.

There are no general guarantees for every equation, but we can do a simple analysis of the error in Euler's method as follows:

First step: \[ y_1 = y_0 + hf(0, y_0) \]

\[ y_1 \approx y(0+h) = y_0 + h \left( y_0 + h \frac{dy}{dt} \bigg|_{t=0} + \frac{h^2}{2!} \frac{d^2y}{dt^2} \bigg|_{t=0} \right) + \ldots \]

\[ y_1 = y_0 + hf(0, y_0) + h^2 \left[ \frac{\text{stuff}}{\text{error}} \right] \]

Assuming the Taylor series is valid on \(0 < t < h\), then the error in the first step is \( E = h^2[\text{stuff}] \) so if we choose \( h \) small enough, the error in one step should shrink like \( h^2 \).
Accumulated error from $t=0$ to $t=T$

We need $\frac{T}{h}$ steps to get from $t=0$ to $t=T$. Each step has error $\sim h^2 \left[ \text{stuff} \right]$. So the accumulated error is estimated as

$$\frac{T}{h} \times h^2 \left[ \text{stuff} \right] \sim h \times \left[ \text{other stuff} \right]$$

If we choose $h$ small enough, we expect the error in our estimate of $y(T)$ to be proportional to $h$. 
Example

\[ y' = -2y + 2 - e^{-4t} \]

\[ y(0) = 1 \]

Estimate \( y(1) \)

Euler's method

\[ y_0 = 1 \]

\[ y_1 = y_0 + h(-2y_0 + 2 - e^{-4t_0}) \]

\[ y_2 = y_1 + h(-2y_1 + 2 - e^{-4t_1}) \]

\[ \vdots \]

up to \( y_N \) where \( N = \frac{1}{h} \).

Analytical solution:

\[ y = \frac{1}{2} e^{-4t} - \frac{1}{2} e^{-2t} + 1 \]