Is this equation EXACT?

\[(2xy + 6) - (x^2 - 3y^2) \frac{dy}{dx} = 0\]

A. YES
B. NO
C. ERRMM?

Is this equation EXACT?

\[(2xy + 6) + (x^2 - 3y^2) \frac{dy}{dx} = 0\]

A. YES
B. NO
C. ERRMM?

Solve the equation:

\[(2xy + 6) + (x^2 - 3y^2) \frac{dy}{dx} = 0\]

A. \((x^2 y) + 6x + (x^2 y - y^3) = C\)
B. \(x^2 y + 6x - y^3 = C\)
C. \(y = -3\)
D. None of the above
E. Cannot be solved

This week’s topics:

- Monday: Exact equations (Lebl 1.8)
- Today: Finish exact equations, introduction to ideas about linear ODE and systems of ODE
- Fri: Euler’s method for numerical solution of ODE

Reading:
- Lebl 1.6 (Autonomous equations, stable vs unstable critical points)
- Lebl 1.8 (Exact)
- Lebl 1.7 (Euler’s method)

For more Examples and an alternative presentation
- Paul’s online notes (linked from class web page) are very helpful
Integrating factor for non-exact equations.

Suppose \( M(x, y) + N(x, y) \frac{dy}{dx} = 0 \) not exact

Multiply by \( u(x, y) \) in the hope of making it exact:

\[
u(x, y) M(x, y) + u(x, y) N(x, y) \frac{dy}{dx} = 0
\]

So we would need:

\[
\frac{\partial u}{\partial y} M + u \frac{\partial M}{\partial y} = \frac{\partial u}{\partial x} N + u \frac{\partial N}{\partial x} \tag{\star}
\]

Generally speaking, solving this for \( u(x, y) \) is no easier than solving the original equation somehow. However, sometimes we can see that \( u = u(x) \) or \( u = u(y) \) would work. There are specific conditions for this given in Lebl 1.8.

For example, if we think that \( u = u(x) \), then \( \frac{\partial u}{\partial y} = 0 \) and (\star) becomes

\[
u \frac{\partial M}{\partial y} = \frac{\partial u}{\partial x} N + u \frac{\partial N}{\partial x}
\]
\[ \frac{\partial u}{\partial x} = \frac{\left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) u}{N(x,y)} \]

\[ u = u(x) \]

so \( \frac{\partial u}{\partial x} = \frac{du}{dx} \)

\[ \frac{du}{dx} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \]

\[ u(x) \]

\[ N(x,y) \]

In order for \( u \) to be just a function of \( x \), we need that \( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \) is only a function of \( x \). If this is true,

\[ \int \frac{1}{u} \frac{du}{dx} \ dx = \int \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \ dx \]

\[ \frac{N(x,y)}{u(x,y)} \]

\[ u = \int \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \ dx \]

\[ e \]

Similar arguments and conditions apply to make \( u = u(y) \) only.
Example \[ \frac{1}{y} + 3y \frac{dy}{dx} = 0 \]

\[
M(x,y) = \frac{1}{y} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2} \]

\[
N(x,y) = 3y \quad \frac{\partial N}{\partial x} = 0 \]

Not exact

Observe: \( M, N \) are only functions of \( y \).
So we'll try an integrating factor \( u(y) \).

\[ u(y) \frac{1}{y} + u(y) 3y \frac{dy}{dx} = 0 \]

Require:

\[
\frac{\partial}{\partial y} \left( u(y) \frac{1}{y} \right) = \frac{\partial}{\partial x} \left( u(y) 3y \right)
\]

\[
\frac{du}{dy} \frac{1}{y} - u \frac{1}{y^2} = 0
\]

\[
\frac{du}{dy} = \frac{u(y)}{y}
\]

\[
\int \frac{1}{u} \frac{du}{dy} dy = \int \frac{1}{y} dy
\]

\[
\ln u = \ln y \quad \boxed{u = y}
\]
Now multiply \( \frac{1}{y} + 3y \frac{dy}{dx} = 0 \) by \( y \):

\[
1 + 3y^2 \frac{dy}{dx} = 0
\]

Observe:

\[
\frac{\partial}{\partial y} (1) = 0 \quad \text{same, now it's exact}
\]

\[
\frac{\partial}{\partial x} (3y^2) = 0
\]

\[
1 + 3y^2 \frac{dy}{dx} = 0
\]

\[
\int \text{wrt } x \quad \downarrow \quad \downarrow \int \text{wrt } y
\]

\[
x \quad y^3
\]

\[
x + y^3 = C
\]

\[
y = (C - x)^{1/3}
\]