Identify the vector field for:

$$\vec{x}'(t) = \begin{pmatrix} 1 & -2 \\ 3 & 3 \end{pmatrix} \vec{x}$$

• A dog is tied to a tree at (0,0). It gets excited and runs around and around the tree with path \(\vec{x}(t) = (x(t),y(t))\), keeping its leash tight at all times. Its leash can slip so it stays the same distance from the tree.
• Is the dog’s motion a solution to an equation of the form \(\vec{x}' = A\vec{x}\) where \(A\) is a constant matrix?
  • A: yes, and \(A\) must have complex eigenvalues
  • B: yes, and \(A\) must have real eigenvalues
  • C: no, this is impossible
  • D: not sure

Announcements:
• Webwork will be posted today; due Wednesday 10pm (happy thanksgiving!)
• HW2 is due on Friday
• HW3 will have two parts:
  – HW 3A will not be collected (but you must do it)
  – HW 3B will be collected in class on Friday, Oct 27
• Midterm 1 will be in class on Oct 18
  – Comprehensive list of topics available next week

This week’s topics:
• Systems of linear equations
• Vector field pictures for 2D autonomous problems
• Eigenvalue/vector method for constant coefficient problems
• Today: summary of features of vector fields, repeated eigenvalue case
• Reading: Lebl 3.1-3.5, 3.7
Defective Matrix Case

- So for, all solutions of $\ddot{\vec{x}} = A\vec{x}$ have been exponentials (or sin/cos, which are very closely related).

  - There are two classes of matrices that do not exactly follow this pattern:
    1. zero eigenvalue (next time)
    2. defective matrix - an n$x$n matrix with less than n linearly independent eigenvectors.

Example: $\ddot{\vec{x}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{x}$

$$r^2 - 2r + 1 = 0$$
$$r - 1)^2 = 0 \text{ repeated eigenvalue } r = 1$$

eigenvectors: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We find only one eigenvector (defective case)

The problem is that I need two solutions but I only have $\vec{x}_1 = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For example, if $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then there is no way to solve using just this.
For this simple case, we can solve directly:

\[ \mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} \]

\[ x_1' = x_1 + x_2 \quad \leftarrow \quad x_2 = c_1 e^t \]

\[ x_2' = x_2 \]

\[ x_1' = x_1 + c_1 e^t \]

\[ x_1' - x_1 = c_1 e^t \quad \text{apply integrating factor } e^{-t} \]

\[ \frac{d}{dt} \left( e^{-t} x_1 \right) = c_1 \]

\[ e^{-t} x_1 = c_1 t + c_2 \]

\[ x_1 = c_1 t e^t + c_2 e^t \]

\[ \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 t e^t + c_2 e^t \\ c_1 e^t \end{pmatrix} \]

\[ = c_1 e^t \begin{pmatrix} t \\ 0 \end{pmatrix} + e^t \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} \]

\[ = c_1 t e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} \]

Observation: \( t e^t \) is useful in the case where we have a defective matrix.