1. Existence and uniqueness for linear problems

\[ \frac{dx}{dt} = a(t)x + b(t)y + f_1(t) \]
\[ \frac{dy}{dt} = c(t)x + d(t)y + f_2(t) \]

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \]

with initial condition \( \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \)

If \( a, b, c, d, f, \) and \( f_2 \) are all continuous on \( \alpha < t_0 < \beta \), then this problem:

(i) has a solution valid on \( \alpha < t_0 < \beta \)

(ii) that solution is unique.
We can always write the solution of this as

\[ \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{x}_H + \vec{x}_p \]

\(\vec{x}_H\) is a solution to \(\vec{x}_H' = A(t) \vec{x}_H\)

(the "homogeneous" problem - no forcing)

Here, we have set the constants using the initial conditions.

\(\vec{x}_p\) is any one solution to \(\vec{x}_p' = A(t) \vec{x}_p + \vec{f}(t)\)

we call this a particular solution.

Observe: \[ \vec{x}' = (\vec{x}_H + \vec{x}_p)' \]

\[ = \vec{x}_H' + \vec{x}_p' \]

\[ = A(t) \vec{x}_H + A(t) \vec{x}_p + \vec{f}(t) \]

\[ = A(t) (\vec{x}_H + \vec{x}_p) + \vec{f}(t) \]

\[ = A(t) \vec{x} + \vec{f}(t) \]

so \(\vec{x}\) is indeed a solution.

Look closely at \(\vec{x}_H\), solution to \(\vec{x}_H' = A(t) \vec{x}_H\).

Superposition

If \(\vec{x}_1\) and \(\vec{x}_2\) are both solutions of the homogeneous problem \(\vec{x}_H' = A(t) \vec{x}_H\), then

\(c_1 \vec{x}_1 + c_2 \vec{x}_2\) is also a solution.
\[ (c_1 \vec{x}_1 + c_2 \vec{x}_2)' \]

\[ = c_1 \vec{x}_1' + c_2 \vec{x}_2' \]

\[ = c_1 A(t) \vec{x}_1 + c_2 A(t) \vec{x}_2 \]

\[ = A(t)(c_1 \vec{x}_1 + c_2 \vec{x}_2) \quad \checkmark \]

**Linear (in)dependence**

\( \vec{x}_1(t) \) and \( \vec{x}_2(t) \) are called **linearly independent** if

\[ c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = 0 \]

for \( c_1 \) and \( c_2 \) not both equal to zero.

(Otherwise **linearly independent**)
Example
Are \( \vec{x}_1(t) = \begin{pmatrix} t \\ e^t \end{pmatrix} \) and \( \vec{x}_2(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix} \) linearly independent?

Yes. We cannot find \( c_1, c_2 \) such that \( c_1 \vec{x}_1 + c_2 \vec{x}_2 = \vec{0} \) valid for all \( t \).

Solution of \( \dot{\vec{x}} = A(t) \vec{x}, \quad \vec{x}(0) = \vec{x}_0 \) is found among the linear combinations \( \vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 \) where \( \vec{x}_1 \) and \( \vec{x}_2 \) are linearly independent solutions.

Applying \( \vec{x}(0) = \vec{x}_0 \),

\[
\begin{align*}
c_1 \, \vec{x}_1(0) + c_2 \, \vec{x}_2(0) &= \vec{x}_0 \\
\begin{pmatrix} \vec{x}_1(0) & \vec{x}_2(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \vec{x}_0
\end{align*}
\]

We call \( \vec{X}(t) = \left( \vec{x}_1(t) \mid \vec{x}_2(t) \right) \) a fundamental matrix for the system of ODEs.

Note: \( \vec{x}_1, \vec{x}_2 \) must be linearly independent solutions.
Initial conditions: \( X(0) \triangleq \hat{c} = \hat{x}_0 \quad \hat{c} = (c_1 \ c_2) \)

Condition for a unique solution is that
\( \det(X(0)) \neq 0 \) or equivalently \( \hat{x}_1, \hat{x}_2 \) are lin. ind.

Fact \( \det(X(t)) \) is either always zero or never zero.

Verify \( \det(X(t)) = \det(\hat{x}, \hat{x}_2) \)
\[ = \det(x_1, x_2) \]
\[ = \det(y_1, y_2) \]
\[ = x_1 y_2 - y_1 x_2 \]

\[ \frac{d}{dt}(\det(X(t))) = x_1' y_2 + x_1 y_2' - y_1' x_2 - y_1 x_2' \]

Recall \((x_i)' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x_i)\)
\[ x_1' = a x_1 + b y_1, \quad y_1' = c x_1 + d y_1, \]
\[ x_2' = a x_2 + b y_2, \quad y_2' = c x_2 + d y_2 \]

\[ \frac{d}{dt}(\det(X(t))) = (a x_1 + b y_1)y_2 + x_1(c x_2 + d y_2) \]
\[ - (a x_2 + b y_2)y_1 - x_2(c x_1 + d y_1) \]
\[
\begin{align*}
&= ax, y_2 + by, y_2 + cx, x_2 + dx, y_2 \\
&\quad - ax_2 y_1 - by_2 y_1 - c x_2 x_1 - d x_2 y_1 \\
&= x_1 y_2 (a+d) - x_2 y_1 (a+d) \\
&= (a+d)(x_1 y_2 - x_2 y_1)
\end{align*}
\]

\[
\frac{d}{dt} \left( \text{det}(X(t)) \right) = (a+d) \text{det}(X(t))
\]

\[
\text{det}(X(t)) = K e^{\int (a+d) \, dt}
\]

- if \( k=0 \)
  \[\text{det}(X(t)) \text{ is always zero}\]
- if \( k \neq 0 \), note \( e^{\int (a+d) \, dt} > 0 \), so \( \text{det}(X) \neq 0 \) ever.

So \( x_1(t) \) and \( x_2(t) \) are either always linearly independent or never linearly independent.

Note: \( \text{det}(X(t)) \) is often called the "Wronskian" but it is often derived differently and looks a little different. (For example, in Paul's online notes) This version is more general.