Which of the following equations has solutions exhibiting resonance:
A. $y'' + 2y = 10 \cos(2t)$
B. $y'' + 4y = 8 \cos(2t)$
C. $y'' + 2y = 6 \cos(4t)$
D. All of the above
E. None of the above

Solutions of $y'' + 100y = 2 \cos(w t)$ display resonance when:
A. $w = 10,000$
B. $w = 10$
C. $w = 9$
D. All of the above
E. None of the above

- Ex: $y' + y = \cos(w t)$
- Solution:
  \[ y = \frac{\cos(t) - \cos(w t)}{w^2 - 1} = \frac{1}{w^2 - 1} \left( \frac{w + 1}{2} \sin \left( \frac{w - 1}{2} t \right) \right) \]

  - $w = 0.8$:

  - $w = 0.95$:

  - $w = 0.99$:

  \[ y = \frac{\cos(t) - \cos(w t)}{w^2 - 1} = 50 \left( 2 \sin \left( 0.975 t \right) \sin \left( 0.005 t \right) \right) \]
Frequency Response (revisited)

\[ LQ'' + RQ' + \frac{1}{c} Q = A \cos \omega t \]

oscillatory forcing

Since we have damping, the homogeneous solution will \( \to 0 \) as \( t \to \infty \). The solution is dominated by \( \phi_Qp \) for large times.

Last time, we calculated \( \phi_Qp \):

\[
\phi_Qp = \frac{A (\cos \omega t) \left(-L\omega^2 + \frac{1}{c}\right)}{(-L\omega^2 + \frac{1}{c})^2 + (R\omega)^2} + \frac{A (\sin \omega t) (-R\omega)}{(-L\omega^2 + \frac{1}{c})^2 + (R\omega)^2}
\]

\[ = D(\omega) \cos \omega t + E(\omega) \sin \omega t \]

\[ = M(\omega) \cos(\omega t - \delta) \quad M^2 = D^2 + E^2 \]

\[ \delta = \arctan \left( \frac{E}{D} \right) \]

\[
(M(\omega))^2 = \frac{A^2 \left(-L\omega^2 + \frac{1}{c}\right)^2 + (-R\omega)^2}{\left[ \left(-L\omega^2 + \frac{1}{c}\right)^2 + (R\omega)^2 \right]^2}
\]

\[ = A^2 \frac{1}{\left(-L\omega^2 + \frac{1}{c}\right)^2 + (R\omega)^2} \]
\[ M(\omega) = \frac{A}{\sqrt{(-L\omega^2 + \frac{1}{c})^2 + (R\omega)^2}} \]

Plot amplitude of the response in the circuit as a function of the forcing frequency.

Find the input frequency that gives the maximum response:

Calculate \( \frac{d(M^2)}{dw} \) and set to zero

\[
\frac{dM^2}{dw} = -A^2 \frac{2(-L\omega^2 + \frac{1}{c})(-2L\omega) + 2R^2\omega}{\left[(-L\omega^2 + \frac{1}{c})^2 + (R\omega)^2\right]^2} = 0
\]

\[ (-L\omega^2 + \frac{1}{c})(-2L\omega) + 2R^2\omega = 0 \]

\[ (-L\omega^2 + \frac{1}{c})(-2L) + R^2 = 0 \]
\[ (-L\omega^2 + \frac{1}{C}) = \frac{R^2}{2L} \]

\[ -L\omega^2 = \frac{R^2}{2L} - \frac{1}{C} \]

\[ \omega_{\text{max}} = \sqrt{\frac{1}{CL} - \frac{R^2}{2L^2}} \]

Observe: if \( R \) is small, \( \omega_{\text{max}} \approx \sqrt{\frac{1}{CL}} \)

which is natural frequency of the circuit with no damping.

response

\[ \sqrt{\frac{1}{CL}} \]

\[ \omega \]
Beat Phenomena

Consider an oscillatory system with no damping.

\[ my'' + ky = \cos \omega t \]
\[ y(0) = y'(0) = 0 \]

Define \( w_0 = \sqrt{\frac{k}{m}} \) = natural frequency.

Suppose \( \omega \neq w_0 \).

Solution: \[ y = C_1 \cos \sqrt{\frac{k}{m}} t + C_2 \sin \sqrt{\frac{k}{m}} t + \frac{A \cos \omega t}{\text{particular}} \]

\[ y = \frac{1}{m(w_0^2 - \omega^2)} (\cos \omega t - \cos w_0 t) \]

\[ \text{Trig: } \cos(A+B) = \cos A \cos B - \sin A \sin B \]
\[ - (\cos(A-B) = \cos A \cos B + \sin A \sin B) \]

\[ \cos(A+B) - \cos(A-B) = -2 \sin A \sin B \]
\[ A + B = \omega t \quad A = \frac{(\omega + w_0)t}{2} \quad B = \frac{(\omega - w_0)t}{2} \]
\[ A - B = w_0 t \]
\[ y = \frac{\delta - 2}{m(w_0^2 - w^2)} \left( \sin \left( \frac{w + w_0}{2} t \right) \right) \sin \left( \frac{w - w_0}{2} t \right) \]

constant amplitude

higher frequency sine

lower frequency sine

If \( w \approx w_0 \) (but \( w \neq w_0 \)) then:

(i) the high frequency sine is oscillating much faster than the low frequency term.

(ii) the amplitude is big

This is called "beat phenomenon."
Summary

1. Resonance \[ y'' + \omega_0^2 y = \cos \omega_0 t \] or \[ \sin \omega_0 t \]

2. Damped system - frequency response
   \[ y'' + by' + \omega_0^2 y = \cos \omega t \]
   
   ![Amplitude vs. Frequency Graph](attachment:image)
   
   (small \( b \))

3. Beats \[ y'' + \omega_0^2 y = \cos \omega t \]
   and \( \omega \approx \omega_0 \)