Summary of Undetermined Coefficients

Method (Guess + Check)

Forcing
- constant
- polynomial of order $n$
- exponential $e^{at}$
- $\sin/cos$

Guess
- constant
- polynomial of order $n$
- exponential $e^{at}$
- $\sin$ and $\cos$

- If the forcing is not linearly independent of the homogeneous solution, try $tx$ normal guess (rarely, $t^2x$)

- Summation of $f(t) + g(t)$

- Summation of relevant guesses.
Resonance

- Forcing an oscillatory system at its natural frequency.

- Classic simple example:

  \[ y'' + y = \cos t \]

  \[ r^2 + 1 = 0 \]

  \[ r = \pm i \]

  \[ y_h = c_1 \cos t + c_2 \sin t \]

  Observe: forcing cost has the same frequency as \( y_h \) \( \leftrightarrow \) not lin. indep.

Guesst:

\[ y_p = \frac{At \cos t + Bt \sin t}{At \cos t - At \sin t + B \sin t + Bt \cos t} \]

\[ y_p' = \frac{-A \sin t - A \sin t - At \cos t + B \cos t + B \cos t}{At \cos t - At \sin t + B \sin t + Bt \cos t} \]

Plug in:

\[ y'' + y = \cos t \]

\[ -2A \sin t + 2B \cos t = \cos t \]

\[ A = 0 \]

\[ 2B = 1 \]

\[ y_p = \frac{1}{2} t \sin t \]
Frequency Response

What is the amplitude of the response in a damped oscillatory system when it is forced by an oscillatory input?

\[ LQ'' + RQ' + \frac{1}{c} Q = V(t) \]

Let \( V(t) = A\cos(wt) \)

\( w = \text{forcing frequency} \)

(1) Homogeneous Solution

\[ Lr^2 + Rr + \frac{1}{c} = 0 \]

\[ r = -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{R^2 - \frac{4L}{c}} \]

Suppose \( R \) is small \( (R^2 < \frac{4L}{c}) \) ⇒ damped oscillations

Let \( r = \lambda \pm i\mu \)

\[ \lambda = -\frac{R}{2L} \quad \mu = \frac{1}{2L} \sqrt{\frac{4L}{c} - R^2} \]

\[ Q(x) = e^{\lambda t} (C_1 \sin(\mu t) + C_2 \cos(\mu t)) \]

\[ Q(x) \rightarrow 0 \text{ as } t \rightarrow \infty \]
(2) **Particular Solution**

Use complex method

\[ L Q''_p + R Q'_p + \frac{1}{c} Q_p = A e^{i\omega t} \]

**Guess:**

\[ Q_p = B e^{i\omega t} \]

\[ Q'_p = i\omega B e^{i\omega t} \]

\[ Q''_p = -\omega^2 B e^{i\omega t} \]

**Plug in:**

\[ B \left( -\omega^2 L + iR\omega + \frac{1}{c} \right) e^{i\omega t} = A e^{i\omega t} \]

\[ Q_p = \text{Re} \left[ \frac{A}{-\omega^2 L + iR\omega + \frac{1}{c}} e^{i\omega t} \right] \]

\[ = \text{Re} \left[ \frac{A \left( \cos \omega t + i\sin \omega t \right)}{-\omega^2 L + iR\omega + \frac{1}{c}} \right] \frac{-\omega^2 L - iR\omega + \frac{1}{c}}{-\omega^2 L - iR\omega + \frac{1}{c}} \]

\[ = A \left( -\omega^2 L + \frac{1}{c} \right) \frac{\cos \omega t}{\left( -\omega^2 L + \frac{1}{c} \right)^2 + (R\omega)^2} \]

\[ + \frac{A \left( R\omega \right)}{\left( -\omega^2 L + \frac{1}{c} \right)^2 + (R\omega)^2} \sin \omega t \]
Let's write in a more convenient form

\[ Q_p = D(w) \cos wt + E(w) \sin wt \]

\[ = M(w) \cos (wt - \delta) \]

\[ M^2 = D^2 + E^2 \]

\[ \delta = \arctan \left( \frac{E}{D} \right) \]

So, as \( t \to \infty \), the circuit has an oscillatory response with frequency \( w \), with amplitude

\[ M(w) = A^2 \frac{(-Lw^2 + \frac{1}{\epsilon})^2 + (Rw)^2}{\left[ (-Lw^2 + \frac{1}{\epsilon})^2 + (Rw)^2 \right]^2} \]

\[ M = \frac{A}{\sqrt{(-Lw^2 + \frac{1}{\epsilon})^2 + (Rw)^2}} \]

Plot \( \frac{M}{A} \)
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- Classic simple example:
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  \[ r = \pm i \]
  \[ y_h = C_1 \cos t + C_2 \sin t \]

Observe: forcing cost has the same frequency as \( y_h \) \( \Leftrightarrow \) not lin. indep.

Guess
  \[ y_p = \frac{A}{2} t \cos t + B t \sin t \]
  \[ y_p' = A \cos t - A t \sin t + B \sin t + B t \cos t \]
  \[ y_p'' = -A \sin t - A \sin t - A t \cos t + B \cos t + B \cos t - B t \sin t \]

Plug in:
  \[ y'' + y = \cos t \]
  \[-2A \sin t + 2B \cos t = \cos t \]
  \[ A = 0 \]
  \[ 2B = 1 \]
  \[ y_p = \frac{1}{2} t \sin t \]
Frequency Response

What is the amplitude of the response in a damped oscillatory system when it is forced by an oscillatory input?

\[ LQ'' + RQ' + \frac{1}{C} Q = V(t) \]

Let \( V(t) = A \cos(\omega t) \)

\( \omega = \text{forcing frequency} \)

(1) Homogeneous Solution

\[ Lr^2 + Rr + \frac{1}{C} = 0 \]

\[ r = -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{R^2 - \frac{4L}{C}} \]

Suppose \( R \) is small \( (R^2 < \frac{4L}{C}) \) \( \Rightarrow \) damped oscillations

Let \( r = \lambda \pm i\mu \)

\( \lambda = -\frac{R}{2L} \)

\( \mu = \frac{1}{2L} \sqrt{\frac{4L}{C} - R^2} \)

\[ Q_1 = e^{\lambda t} (C_1 \sin(\mu t) + C_2 \cos(\mu t)) \]

\[ Q_1 \rightarrow 0 \text{ as } t \rightarrow \infty \]
(2) **Particular Solution**

\[ L Q'' + R Q' + \frac{1}{c} Q = A e^{iwt} \]

**Guess:**
\[
Q_p = Be^{iwt} \\
Q'_p = i\omega Be^{iwt} \\
Q''_p = -\omega^2 Be^{iwt}
\]

**Plug in:**
\[
B \left( -\omega^2 L + iRw + \frac{1}{c} \right) e^{iwt} = A e^{iwt}
\]

\[
Q_p = \text{Re} \left[ \frac{A}{-\omega^2 L + iRw + \frac{1}{c}} e^{iwt} \right]
\]

\[
= \text{Re} \left[ \frac{A(\cos \omega t + i \sin \omega t)}{-\omega^2 L + iRw + \frac{1}{c}} \right] \frac{-\omega^2 L - iRw + \frac{1}{c}}{-\omega^2 L - iRw + \frac{1}{c}}
\]

\[
= A \left( -\frac{\omega^2 L + \frac{1}{c}}{(-\omega^2 L + \frac{1}{c})^2 + (Rw)^2} \right) \cos \omega t
\]

\[
+ A \left( \frac{Rw}{(-\omega^2 L + \frac{1}{c})^2 + (Rw)^2} \right) \sin \omega t
\]
Let's write in a more convenient form
\[
Q_p = D(w) \cos \omega t + E(w) \sin \omega t
\]
\[
= M(w) \cos(\omega t - \delta)
\]
\[
M^2 = D^2 + E^2
\]
\[
\delta = \arctan\left(\frac{E}{D}\right)
\]
So, as \(t \to \infty\), the circuit has an oscillatory response with frequency \(\omega\), with amplitude
\[
(M(w))^2 = A^2 \frac{\left(-L\omega^2 + \frac{1}{c}\right)^2 + (R\omega)^2}{\left[\left(-L\omega^2 + \frac{1}{c}\right)^2 + (R\omega)^2\right]^2}
\]
\[
M = \frac{A}{\sqrt{\left(-L\omega^2 + \frac{1}{c}\right)^2 + (R\omega)^2}}
\]
Plot \(\frac{M}{A}\)