Instructions

• The total time allowed is 50 minutes.

• The total score is 40 points.

• Use the reverse side of each page if you need extra space.

• Show all your work. A correct answer without intermediate steps will receive no credit.

• Calculators, phones and cheat sheets are not allowed.

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1. (14 points)

a) (5 points) Find a general form of homogeneous solutions for the equation

\[ \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{x}(t). \]

Solution:
First compute the eigenvalues of the matrix.

\[ \det \left( \begin{bmatrix} 3 - \lambda & -2 \\ -1 & 4 - \lambda \end{bmatrix} \right) = (3 - \lambda)(4 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 \]

\( (\lambda - 2)(\lambda - 5), \lambda_1 = 2, \lambda_2 = 5. \)
Next compute the eigenvectors. \( \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \)
\( \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \)
The general form of homogeneous solutions is a linear combination of \( e^{\lambda_i} \vec{v}_i. \)

\[ \vec{x}_H(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

b) (2 points) Circle the best description of the equilibrium \( \vec{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) for the system in a).
Solution: Both eigenvalues are real and positive so the origin is an
Unstable node;

c) (7 points) Find a particular solution to

\[ \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 2t \\ t \end{bmatrix}. \]

Solution 1: Use undetermined coefficients. The general guess of particular solution is

\[ \vec{x}_p(t) = \begin{bmatrix} A_1t \\ A_2t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \]
Plugging this into the differential equation gives two \( 2 \times 2 \) linear systems

\[ \frac{d}{dt} \vec{x}_p(t) - \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{x}_p(t) = \begin{bmatrix} A_1 - 3B_1 + 2B_2 \\ A_2 + B_1 - 4B_2 \end{bmatrix} + \begin{bmatrix} -3A_1t + 2A_2t \\ A_1t - 4A_2t \end{bmatrix}. \]
First we solve the system matching the coefficients of \( t \)

\[ \begin{bmatrix} -3A_1 + 2A_2 \\ A_1 - 4A_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]
This is easy to solve because \[
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\] is an eigenvector with eigenvalue \(-2\) (the minus because this is the negative of the matrix from a). The solution is \(A_1 = -1\) and \(A_2 = -1/2\). The second equation is

\[
\begin{bmatrix}
-3B_1 + 2B_2 \\
B_1 - 4B_2
\end{bmatrix} = \begin{bmatrix}
1 \\
1/2
\end{bmatrix},
\]

which is the same but divided by two, so \(B_1 = -1/2, B_2 = -1/4\).

**Solution 2:** Use the variation of parameters formula

\[
\vec{x}_P(t) = X(t) \int_0^t X^{-1}(s) \begin{bmatrix} 2s \\ s \end{bmatrix} ds
\]

where the fundamental matrix comes from the general homogeneous solution and is

\[
X(t) = \begin{bmatrix} 2e^{2t} & e^{5t} \\
e^{2t} & -e^{5t}
\end{bmatrix}.
\]

The determinant is \(\det(X(t)) = -3e^{7t}\), so the inverse is

\[
X^{-1}(s) = \frac{1}{-3e^{7s}} \begin{bmatrix} -e^{5s} & -e^{5s} \\ -e^{2s} & 2e^{2s}
\end{bmatrix}.
\]

Next do a matrix-vector multiplication

\[
X^{-1}(s) \begin{bmatrix} 2s \\ s \end{bmatrix} = \frac{1}{-3e^{7s}} \begin{bmatrix} -3se^{5s} \\ 0
\end{bmatrix} = \begin{bmatrix} se^{-2s} \\ 0
\end{bmatrix}.
\]

Now integrate by parts to get

\[
\int_0^t X^{-1}(s) \begin{bmatrix} 2s \\ s \end{bmatrix} ds = -\frac{1}{2} \begin{bmatrix} te^{-2t} \\ 0
\end{bmatrix} - \frac{1}{4} \begin{bmatrix} e^{-2t} \\ 0
\end{bmatrix}.
\]

One final matrix-vector multiplication yields

\[
X(t) \int_0^t X^{-1}(s) \begin{bmatrix} 2s \\ s \end{bmatrix} ds = -\frac{1}{2} \begin{bmatrix} 2t \\ t
\end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 \\ 1
\end{bmatrix}.
\]

**Solution:**

\[
\vec{x}_P(t) = \begin{bmatrix} -t \\ -t/2
\end{bmatrix} + \begin{bmatrix} -1 \\ -1/2
\end{bmatrix}.
\]
2. (12 points) A damped spring with mass 1 kg has damping constant \( \alpha \) kg/s and spring constant 4 kg/s\(^2\).

a) (4 points) For what values of \( \alpha \geq 0 \) is the spring under-damped, over-damped, and critically-damped? (Repeat this question for a circuit with resonance on 2nd version?)

**Solution:** The damped spring-mass equation is

\[
\ddot{x}(t) + \alpha \dot{x}(t) + 4x(t) = 0.
\]

The characteristic polynomial is \( \lambda^2 + \alpha \lambda + 4 = 0 \), which has roots

\[
\lambda = -\frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha^2 - 16}.
\]

Over-damped occurse when the roots are real and distinct, underdamped if they are complex, and critical if they are real and repeated.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Condition</th>
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<tr>
<td>( \alpha = 4 )</td>
<td>Critical case</td>
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<tr>
<td>( \alpha &gt; 4 )</td>
<td>Over-damped</td>
</tr>
<tr>
<td>( \alpha &lt; 4 )</td>
<td>Under-damped</td>
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b) (4 points) For \( \alpha = 0 \) and a force applied of 3sin(3t) kg m/s\(^2\), compute a particular solution of the undamped spring-mass equation.

**Solution:**

Guess a particular solution of the form \( x_P(t) = A \sin(3t) \), the equation is

\[
-9A \sin(3t) + 4A \sin(3t) = 3 \sin(3t),
\]

which is solved by \( A = \frac{3}{5} \).

**Solution:** \( x_P(t) = \frac{3}{5} \sin(3t) \)

c) (2 points) Solve the equation in b) for the initial condition \( x(0) = 1, \ x'(0) = 1 \).

**Solution:**

The homogeneous solution is \( c_1 \cos(2t) + c_2 \sin(2t) \). The initial position implies \( c_1 = 1 \). To compute the value of \( c_2 \) solve \( 2c_2 + 3(\frac{-3}{5}) = 1 \). \( 2c_2 = \frac{14}{5}, \ c_2 = \frac{7}{5} \).

**Solution:** \( x(t) = \cos(2t) + \frac{7}{5} \sin(2t) - \frac{3}{5} \sin(3t) \)
3. (14 points) The following questions concern the equation

\[ x'' + 2x' + 2x = 3. \]

a) (2 points) Express the equation as a system of equations for \( \vec{y} = \begin{bmatrix} x \\ x' \end{bmatrix} \)

Solution:

\[
\frac{d}{dt} \vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}
\]

b) (8 points) Let \( A \) be the matrix form part a). Compute \( \exp(tA) \).

Hint: it is much simpler to find the exponential by first computing the fundamental matrix, rather than using diagonalization.

Solution:

Compute the eigenvalues:

\[-\lambda(-\lambda - 2) + 2 = \lambda^2 + 2\lambda + 2, \lambda = -1 \pm \frac{1}{2}\sqrt{4 - 8} = -1 \pm i.\]

The fundamental matrix is formed from the general homogeneous solution

\[ x_H(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) \]

and its derivatives:

\[ X(t) = \begin{bmatrix} e^{-t} \cos(t) & e^{-t} \sin(t) \\ -e^{-t} \cos(t) - e^{-t} \sin(t) & -e^{-t} \sin(t) + e^{-t} \cos(t) \end{bmatrix}. \]

For the matrix exponential we need to correct so the initial value is the identity.

\[ X(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \]

\[ X(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]

\[ X(t)X(0)^{-1} = \begin{bmatrix} e^{-t} \cos(t) + e^{-t} \sin(t) & e^{-t} \sin(t) \\ -2e^{-t} \sin(t) & -e^{-t} \sin(t) + e^{-t} \cos(t) \end{bmatrix} \]

Solution:

\[ e^{At} = \begin{bmatrix} e^{-t} \cos(t) + e^{-t} \sin(t) & e^{-t} \sin(t) \\ -2e^{-t} \sin(t) & -e^{-t} \sin(t) + e^{-t} \cos(t) \end{bmatrix} \]

c) (4 points) Solve for \( x(t) \) with the initial conditions

\[ x(0) = 0, \quad x'(0) = 1. \]

Solution:

The particular solution is \( x_P(t) = \frac{3}{2} \). The solution is given by

\[ \vec{y}(t) = e^{At} \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}. \]

Solution:

\[ x(t) = \frac{-3}{2} e^{-t} \cos(t) - \frac{1}{2} e^{-t} \sin(t) + \frac{3}{2}. \]