Math 215/255 A1
Midterm 1, October 18, 2017

Name: 			 SID:
Instructor: 		 Section:

Instructions

• The total time allowed is 50 minutes.
• The total score is 40 points.
• Use the reverse side of each page if you need extra space.
• Show all your work. A correct answer without intermediate steps will receive no credit.
• Calculators, phones and cheat sheets are not allowed.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>7</td>
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<td>TOTAL</td>
<td>40</td>
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</tbody>
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1
1. (7 points) Solve the following initial value problems for $y(t)$:

$$ty' + (1 + t)y = 2te^{-2t}, \ y(1) = 0.$$
2. (7 points) Consider the following system of first order ODEs
\[
\begin{align*}
\frac{dy_1}{dt} &= 3y_1(t) + y_2(t) \\
\frac{dy_2}{dt} &= 2y_1(t) + 2y_2(t)
\end{align*}
\]
(a) Find the general solution of the system.

Solution:

(b) Use the initial conditions \( y_1(0) = 2 \) and \( y_2(0) = -2 \) to find the constants in your solution.

Solution:
3. (8 points) Consider the following initial value problem:

\[ \frac{dy}{dx} = \frac{-xy^3}{(x^2y^2 + 1)}, \quad y(1) = 1 \]

(a) Is the differential equation exact?

Solution:

(b) If YES, find a solution that satisfies the given initial condition. Otherwise, find an integrating factor \( h(y) \) and use it to find a solution to the equation that satisfies the given initial condition.

Solution:
4. (6 points) Consider the slope fields below.

(i) The slope fields indicates that the associated differential equation is of which form?

(a) \( y'(t) = f(t) \)
(b) \( y'(t) = f(y) \)
(c) \( y'(t) = f(t, y) \)
(d) None of the above

(ii) Given the initial condition \( y(0) = 1 \), use the slope field to determine the value of \( y(t) \) as \( t \to \infty \)?

Solution: 

5
(iii) Which of the following Matlab commands will produce this direction field

(a)

```matlab
f = @(t,y) (y-3)/t; % defining the function f(t,y)
dirfield(f, -5:.3:5, -5:0.3:5); % plotting the direction field
title('Slope field');
xlabel('time')
ylabel('y(t)')
```

(b)

```matlab
f = @(t,y) t*(t-3); % defining the function f(t,y)
dirfield(f, -5:.3:5, -5:0.3:5); % plotting the direction field
title('Slope field');
xlabel('time')
ylabel('y(t)')
```

(c)

```matlab
f = @(t,y) 0.1*y*(y^2-9)*t; % defining the function f(t,y)
dirfield(f, -5:.3:5, -5:0.3:5); % plotting the direction field
title('Slope field');
xlabel('time')
ylabel('y(t)')
```

(d)

```matlab
f = @(t,y) 0.1*y*(y^2-9); % defining the function f(t,y)
dirfield(f, -5:.3:5, -5:0.3:5); % plotting the direction field
title('Slope field');
xlabel('time')
ylabel('y(t)')
```
5. (12 points) The population of a certain bacteria grows according to the following differential equation
\[
\frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P, \quad \text{where } r, K > 0
\]
(a) Explain the meaning of the parameters \( r \) and \( K \).

(b) Find all the equilibria (steady state solutions) of the differential equation.

Solution:

(c) Sketch the graph of \( \frac{dP}{dt} \) vs \( P \) and use it to determine which of these equilibria is stable, unstable, or semi-stable.

Solution:
(d) Use the initial condition \( P(0) = P_0 \) \( (P_0 > 0) \) to find the population \( P(t) \) of the bacteria. Hint: You may need the partial fraction

\[
\frac{1}{P(P-a)} = \frac{A}{P} + \frac{B}{(P-a)}.
\]

Solution:

(e) What is the population \( P(t) \) as \( t \rightarrow \infty \)?

Solution: