Math 215/255
Midterm 1, February 17, 2017

Name:  
SID:  
Instructor:  
Section:  

Instructions

• The total time allowed is 50 minutes.
• The total score is 40 points.
• Use the reverse side of each page if you need extra space.
• Show all your work. A correct answer without intermediate steps will receive no credit.
• Calculators, phones and cheat sheets are not allowed.

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1. (12 points) Solve the following initial value problems for $y(t)$:

(a) $y' - 4t^2 = -t^2 y$, $y(0) = 8$.

\[
\frac{dy}{dt} = t^2 (4 - y), \quad \text{Separable}
\]

\[
\frac{dy}{4 - y} = t^2 \, dt
\]

\[
\int \frac{dy}{4 - y} = \int t^2 \, dt + C \quad \implies \quad -\ln |4 - y| = \frac{1}{3} t^3 + C
\]

\[
4 - y = C_1 e^{-\frac{1}{3} t^3}, \quad y(0) = 8 \quad \implies \quad -4 = C_1
\]

Solution: $y = 4 + 4e^{-\frac{1}{3} t^3/3}$

(b) $y' = -\frac{x(3xy + y^2)}{x(x^2 + xy)}$, $y(1) = 2$.

\[
\frac{x^2}{x^2 + xy} \, dy + \frac{x(3xy + y^2)}{x(x^2 + xy)} \, dx = 0
\]

$M(x, y)$

$N(x, y)$

\[
\frac{\partial M}{\partial x} = x^2 + xy + x(2x + y) = 3x^2 + 2xy
\]

\[
\frac{\partial N}{\partial y} = x(3x + 2y) = 3x^2 + 2xy
\]

\[
\text{exact equation}
\]

Potential function: $M = \frac{\partial \Phi}{\partial y}$, $N = \frac{\partial \Phi}{\partial x}$

Solution: $\frac{x^3 y + \frac{x^2 y^2}{2}}{2} = 4$
1b) Cont'd:

\[ \phi = \int M \, dy + h(x) \]

\[ = \int \left( x^3 + x^2 y \right) \, dy + h(x) \]

\[ = xy^3 + \frac{x^2 y^2}{2} + h(x) \]

\[ \frac{d\phi}{dx} = N(x) \]

\[ 3xy^2 + xy^2 + \frac{dh(x)}{dx} = 3x^2 y + xy^2 \]

\[ \frac{dh(x)}{dx} = 0 \Rightarrow h(x) = C \]

\[ \phi = xy^3 + \frac{x^2 y^2}{2} + C \]

Solution:

\[ \phi = C_1 \text{ Constant} \Rightarrow x^3 + \frac{x^2 y^2}{2} = C_1 \]

\[ y(1) = 2 \]

\[ 2 + \frac{4}{2} = C_1 \Rightarrow C_1 = 4 \]
2. (6 points) Solve the initial value problem for $\vec{x}(t)$:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 6 \\ 0 \end{pmatrix}. $$

$$\det \begin{pmatrix} 2 - \lambda & 2 \\ -2 & 2 - \lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (2 - \lambda)^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{4}{2} \pm \sqrt{16 - 32} / 2 = 2 \pm 2i$$

$$\lambda_1 = 2 + 2i, \quad \text{Find the eigenvector:}$$

$$\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -2v_1i + 2v_2 = 0 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{X}_1 = e^{2t \begin{pmatrix} \cos 2t + i \sin 2t \\ -\sin 2t + i \cos 2t \end{pmatrix}} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2t} \begin{pmatrix} \cos 2t + i \sin 2t \\ -\sin 2t + i \cos 2t \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$
2) cont'd:

\[ \vec{X}_g = C_1 e^{2t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} \]

\[ \vec{X}(0) = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} 6 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \Rightarrow C_1 = 6, \quad C_2 = 0 \]

\[ \vec{X} = 6 e^{2t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} \]
3. (6 points) Consider the initial value problem $y^3y' = x^3$, $y(0) = 0$.

(a) This problem has solutions $y_1(x) = x$ and $y_2(x) = -x$. Does this contradict our knowledge about existence and uniqueness? Explain.

\[
y' = \frac{x^3}{y^3} = \frac{f(x,y)}{y^3}, \quad f(x,y) \text{ and } \frac{\partial f}{\partial y} \text{ are not continuous at } y=0. \text{ So the theorem does not guarantee uniqueness.}
\]

(b) Solve the related problem $y^3y' = x^3$, $y(0) = 1$.

\[
\int y^3 \, dy = \int x^3 \, dx + C
\]
\[
\frac{1}{4} y^4 = \frac{1}{4} x^4 + C
\]
\[
y(0) = 1 \rightarrow C = \frac{1}{4}
\]
\[
y^4 = x^4 + 1
\]
\[
y = \left( x^4 + 1 \right)^{1/4}
\]

(Note: the negative sign does not satisfy the I.C. and is not acceptable.)
4. (6 points) Consider the initial value problem \( \frac{dy}{dt} = y(y-1)^2, \ y(0) = y_0. \)

(a) Find all possible equilibrium solutions.

\[ \frac{dy}{dt} = 0 \Rightarrow y = 0, \ y = 1 \]

**Solution:**

\( y = 0, \ y = 1 \)

(b) Find \( \lim_{t \to \infty} y(t) \) for all possible values of \( y_0 \) between \(-\infty\) and \(\infty\).

\[ y_0 < 0: \lim_{t \to \infty} y(t) = -\infty \]

\[ y_0 = 0: \lim_{t \to \infty} y(t) = 1 \]

\[ y_0 > 0: \lim_{t \to \infty} y(t) = \infty \]

**Solution:**

or: \( t = \infty \)

semi-stable

unstable
5. (4 points) The following Matlab script plots the phase plane for a linear system \( \dot{\mathbf{x}} = A\mathbf{x} \) and uses Matlab's ode45 to solve a particular solution for a given initial condition and plot this solution on the phase portrait.

\[
A = [-3 1; 1 1];
\]

\[
[x, y] = \text{meshgrid}([-5:0.5:5,-5:0.5:5]);
\]

\[
xp = A(1,1)\times x + A(1,2)\times y;
\]

\[
yp = A(2,1)\times x + A(2,2)\times y;
\]

\[
\text{quiver}(x,y,xp,yp); \text{hold on} ;
\]

\[
f = @(t,x) A\times x;
\]

\[
ic = [3; 0];
\]

\[
t = [0 50];
\]

\[
[tout, xout] = \text{ode45}(f, t, ic);
\]

\[
\text{plot}(xout(:,1), xout(:,2), 'r', 'LineWidth', 3);
\]

\[
\lambda^2 + 2\lambda - 4 = 0
\]

\[
\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{4 + 16}}{2} = -1 \pm \sqrt{5}
\]

\[
\lambda_1 > 0, \lambda_2 < 0
\]

(a) Write the equations for the initial value problem that this code solves.

\[
\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}
\]

(b) Circle the figure that is generated by the code above. No work need be shown.
6. (6 points) The initial value problem:

\[ y' = y + t; \quad y(0) = 0, \]

is solved in the following Matlab script using Euler's method.

```matlab
N=2; % choose number of Euler steps
% set start and end times for the problem
endtime = 0.2;
starttime = 0;
initial_condition = 0; % this should be y(starttime) for the problem

% RHS function (set to an example y'=y+t)
f=@(t,y) t+y;

% define the step size h
h=(endtime-starttime)/N;

% now set the first elements of y_euler and t equal to their starting values
y_euler(1)=initial_condition;
t(1)=starttime;

% this loop implements the Euler method described in class
for k=1:N
    t(k+1)=t(k)+h;
    y_euler(k+1)=y_euler(k)+h*f(t(k),y_euler(k));
end

y_exact=@(t) 1*exp(t)-1-t;
errorglobal=abs(y_euler(N+1)-y_exact(endtime));
```

(a) Write down the algebraic formula that is used to integrate the equation numerically.

\[ y(t_n+h) = y(t_n) + h \cdot f(y_n, t_n) \]

\[ y(t_0) = y_0 \]

(b) What is the step size?

\[ h = \frac{0.2}{2} = 0.1 \]

(c) What is the estimate of \( y \) at \( t = 0.2 \) that would be produced by this code?

\[ y(0) = 0 \]

\[ y(0.1) = 0 + 0.1(0 + 0) = 0 \]

\[ y(0.2) = 0 + 0.1(0.1 + 0) = 0.01 \]