HOMEWORK 5 SOLUTIONS: MATH 215 Winter 2017

1. The functions sinh(t) and cosh(t) are defined by

\[ \sinh(t) = \frac{1}{2} (e^t - e^{-t}) , \quad \cosh(t) = \frac{1}{2} (e^t + e^{-t}) . \]

You should check that you agree that sinh(0) = 0, cosh(0) = 1, cosh(t) = cosh(-t), sinh(t) = -sinh(-t) and

\[ \frac{d}{dt} \sinh(t) = \cosh(t) , \quad \frac{d}{dt} \cosh(t) = \sinh(t) . \]

- (a) Verify (by plugging in) that \( y = e^{-kt} \), \( y = e^{kt} \) and \( y = A \sinh(k(t - t_0)) + B \cosh(k(t - t_0)) \) are solutions of \( y'' - k^2 y = 0 \) for all \( t > 0 \) and for all constants \( k > 0, t_0, A \) and \( B \).
- (b) Using sinh and cosh, find the solution \( y = y(t) \) to the following initial value problems:

\[
\begin{align*}
 y'' - 16y &= 0, \quad y(0) = 1, \quad y'(0) = 0 \\
 y'' - 9y &= 0, \quad y(3) = 1, \quad y'(3) = 0
\end{align*}
\]

For the second problem it is more convenient to look for a solution in the form \( y = A \sinh(k(t-t_0)) + B \cosh(k(t-t_0)) \) where \( k = 3 \) and \( t_0 = 3 \).

Solution:

- (a) Let \( y = e^{\pm kt} \), then \( y' = \pm ke^{\pm kt} \) and \( y'' = k^2 e^{\pm kt} \). Hence \( y'' - k^2 y = 0 \) is satisfied. Now since \( \sinh(k(t-t_0)) \) and \( \cosh(k(t-t_0)) \) are both linear combinations of \( e^{kt} \) and \( e^{-kt} \), it follows by the superposition principle that they are both solutions.

- (b) The solution to \( y'' - 16y = 0 \) can be written as \( y = c_1 \sinh(4t) + c_2 \cosh(4t) \). Now, from the properties of sinh(t) and cosh(t) given above, we get \( c_2 = 1 \) and \( c_1 = 0 \). Thus, the solution is \( y = \cosh(4t) = \frac{1}{2} (e^{4t} + e^{-4t}) \). The solution to \( y'' - 9y = 0 \) can be written as \( y = c_1 \sinh(3(t-3)) + c_2 \cosh(3(t-3)) \). Now, from the properties of sinh(t) and cosh(t) given above, we get \( c_2 = 1 \) and \( c_1 = 0 \). Thus, the solution is \( y = \cosh(3(t-3)) = \frac{1}{2} (e^{3(t-3)} + e^{-3(t-3)}) \).

2. Charging of a capacitor: Consider the RLC circuit, where the resistor, inductor and capacitor are in series with an external D.C. voltage source which is active for \( 0 < t < T \) but is then disconnected from the circuit at \( t = T \). The fixed time \( T \) is called the switching time. Then, the charge \( Q(t) \) on the capacitor satisfies

\[ LQ'' + RQ' + \frac{1}{C}Q = \begin{cases} V_0 & \text{for } 0 \leq t < T, \\
0 & \text{for } t \geq T, \end{cases} \]

where \( V_0 \) is a constant. Assume that \( Q(0) = Q'(0) = 0 \), and that \( R^2 > 4L/C \). Find the solution \( Q = Q(t) \) and sketch by hand two plots of \( Q(t) \), one for “large” \( T \) and one for “small” \( T \). Explain the result on physical terms.

Solution: Let \( Q = e^{rt} \) in the homogeneous problem to get \( Lr^2 + Rr + 1/C = 0 \), so that

\[ r_{\pm} = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4L/C}}{2L} . \]
The roots are real and negative since \( R^2 > 4L/C \). The particular solution is \( Q = CV_0 \) for \( 0 < t < T \). Hence, we have

\[
Q(t) = \begin{cases} 
  c_1 e^{r_+ t} + c_2 e^{r_- t} + CV_0 & \text{for } 0 \leq t < T, \\
  d_1 e^{r_+(t-T)} + d_2 e^{r_-(t-T)} & \text{for } t > T.
\end{cases}
\]

(1)

There are four constants to determine: \( c_1, c_2, d_1 \) and \( d_2 \). Satisfying the initial conditions we get two equations

\[
c_1 + c_2 + CV_0 = 0 \quad \text{and} \quad r_+ c_1 + r_- c_2 = 0.
\]

Thus,

\[
c_2 = -\frac{r_+ CV_0}{(r_+ - r_-)}, \quad c_1 = \frac{r_- CV_0}{(r_+ - r_-)}, \quad \text{where} \quad r_+ - r_- = \frac{\sqrt{(R^2 - 4L/C)}}{L}.
\]

(2)

Now we satisfy the conditions that \( Q \) and \( Q' \) are continuous at \( t = T \). This yields two equations for \( d_1 \) and \( d_2 \). We get

\[
d_1 + d_2 = c_1 e^{r_+ T} + c_2 e^{r_- T} + CV_0, \quad r_+ d_1 + r_- d_2 = c_1 r_+ e^{r_+ T} + c_2 r_- e^{r_- T}.
\]

We can solve for \( d_1 \) and \( d_2 \) to get

\[
d_1 = c_1 e^{r_+ T} - \frac{CV_0 r_-}{(r_+ - r_-)}, \quad d_2 = c_2 e^{r_- T} + \frac{CV_0 r_+}{(r_+ - r_-)}.
\]

(3)

Substituting (2) and (3) into (1) gives \( Q = Q(t) \). Notice that when \( T \) is large (i.e. \( r_+ T \gg 1 \) and \( r_- T \gg 1 \)), then we can neglect the exponential terms in (3). In addition, the exponential terms in the formula for \( Q(t) \), defined on the interval \( 0 \leq t < T \), are insignificant when \( t \approx T \). Thus, the capacitor has had time to be almost fully charged and \( Q \approx CV_0 \) when \( t \approx T \). The plots of the solutions are shown below in Fig. 1 for \( T \) small (the capacitor is not fully charged) and for \( T \) large when the capacitor is fully charged. In both cases when \( t > T \), the charge on the capacitor leaks out as \( t \to \infty \) since the circuit has resistance.

![Figure 1: Plot of \( q(t) \) versus \( t \) for \( T = 5 \) (heavy solid curve) and \( T = 20 \) (solid curve). The parameters are \( C = V_0 = L = 1 \) and \( R = \sqrt{20} \).](image)

3. Solve \( y'' + 25y = \cos(5.6t) \) with \( y(0) = 0 \) and \( y'(0) = 0 \). Use Matlab/Octave (or similar) to make an accurate plot of the solution. What do you observe?
Solution: For the homogeneous problem let \( y = e^{rt} \). Then, \( r^2 + 25 = 0 \) so that \( r = \pm 5i \). The solution to the homogeneous problem is \( y = \sin(5t) \) or \( y = \cos(5t) \). Now calculate the particular solution. Let \( \tilde{y} \) solve
\[
\tilde{y}'' + 25\tilde{y} = e^{i\omega t}, \quad \omega = 5.6
\]
Then \( y = \text{Re}(\tilde{y}) \). Let \( \tilde{y} = Ae^{i\omega t} \). Substitute to get \( A(-\omega^2 + 25) = 1 \). Thus, \( A = 1/(25 - \omega^2) \). Thus, the particular solution is
\[
y_p = \text{Re} \left( (25 - \omega^2)^{-1} e^{i\omega t} \right) = \frac{\cos(\omega t)}{25 - \omega^2}.
\]
The general solution is
\[
y = c_1 \cos(5t) + c_2 \sin(5t) + \frac{\cos(\omega t)}{25 - \omega^2}.
\]
Satisfying the initial conditions \( y(0) = y'(0) = 0 \), we get \( c_2 = 0 \), \( c_1 = -1/(25 - \omega^2) \). Thus, using a trig identity we can write
\[
y = \frac{1}{(25 - \omega^2)} [\cos(\omega t) - \cos(5t)] = \frac{1}{3.18} \sin(0.3t) \sin(5.3t).
\]
The solution, which exhibits the phenomenon of beats as described in class, is plotted below in Fig. 2.

![Figure 2: Plot of \( y(t) = \frac{10}{3.18} \sin(0.3t) \sin(5.3t) \) versus \( t \).](image)

4. (This is an Important Problem:) Consider \( y'' + py' + y = F_0 \sin(\omega t) \), where \( p > 0 \) and \( F_0 \) are constants. Use Matlab/Octave (or similar) to plot the amplitude of the steady-state (long-term) response as a function of \( \omega \) on the interval \( \omega > 0 \). Make these plots for several representative values of \( p \). Where is the amplitude a maximum as a function of \( \omega \)?

Solution: Let \( \tilde{y} \) satisfy
\[
\tilde{y}'' + p\tilde{y}' + \tilde{y} = F_0 e^{i\omega t}.
\]
Then $y_p = \text{Im}(\tilde{y})$. Substitute $\tilde{y} = Ae^{i\omega t}$ into (1) to get

$$A \left[ (1 - \omega^2) + i p \omega \right] = F_0,$$

$$\rightarrow A = \frac{F_0}{(1 - \omega^2) + i p \omega}.$$ 

Thus,

$$y_p = \text{Im} \left[ F_0 \frac{(1 - \omega^2) - i p \omega}{(1 - \omega^2)^2 + p^2 \omega^2} (\cos(\omega t) + i \sin(\omega t)) \right].$$

A simple calculation then yields

$$y_p = R(\omega) \cos(\omega t - \phi(\omega)),$$

where $R(\omega)$ and $\phi(\omega)$ satisfy

$$R(\omega) = \frac{F_0}{\left[ (1 - \omega^2)^2 + p^2 \omega^2 \right]^{1/2}}, \quad \tan(\phi) = \frac{1 - \omega^2}{p \omega}, \quad \text{(*)}$$

with $\cos(\phi) < 0$. The amplitude of the steady-state response is $R(\omega)$. It has a maximum, on the interval $\omega \geq 0$, when the denominator in (*) has a minimum on this interval. Using a little calculus, we find that this occurs at $\omega = \omega_c$, where

$$\omega_c = \sqrt{1 - \frac{p^2}{2}}, \quad 0 < p < \sqrt{2}; \quad \omega_c = 0, \quad p \geq \sqrt{2}.$$ 

Thus, if the damping coefficient $p$ is not too large (i.e. $p < \sqrt{2}$), then the system has its largest response at a frequency $\omega_c \neq 0$. The maximum amplitude $R(\omega_c)$ increases as $p$ decreases and becomes unbounded as $p \to 0$. The frequency $\omega_c$ tends to the resonant frequency of 1 as $p \to 0$. When $F_0 = 1$, we plot $R(\omega)$ versus $\omega$ below in Fig. 3 for $p \approx 0$, for $p = 1$, $p = \sqrt{2}$ and $p > \sqrt{2}$.

![Figure 3: Plot of $R(\omega)$ in (*) versus $\omega$ for $p = 0.2$ (heavy solid curve), $p = 1.0$ (solid curve), $p = \sqrt{2}$ (dotted curve), and $p = 2$ (widely spaced dots).](image-url)
5. (Tuning a circuit) Consider an \( RLC \) circuit in series with an A.C. voltage source given by \( V(t) = -\cos t - 4/5 \cos(5t) \). Suppose that \( R = .1 \) Ohms, \( L = 1 \) Henry but that we are capable of adjusting the capacitance \( C \) of the capacitor.

- (i) Show that the current in the circuit satisfies
  \[
  I'' + 0.1I' + \frac{1}{C}I = \sin t + 4 \sin(5t)
  \]

- (ii) Calculate the steady-state (long-term) solution. This is the solution after any transient terms have decayed to zero.

- (iii) What are the critical values of \( C \) for which the steady-state current \( I \) will (roughly) be periodic with a frequency of either 1 or 5? Interpret this result in terms of the tuning of a circuit.

- (iv) Use Matlab/Octave (or similar) to plot the steady-state solution when \( C = 1 \), \( C = 1/25 \) and \( C = 1/81 \).

Solution:

- (i) The voltage drops across the three elements is equal to the applied voltage. This yields,
  \[
  LQ' + RQ + Q/C = -\cos t - \frac{4}{5} \cos(5t),
  \]
  where \( Q \) is the charge on the capacitor. Now let \( L = 1 \) Henry, \( R = .1 \) Ohm and note that \( I = Q' \), where \( I \) is the current flowing through the circuit. Thus, upon differentiating (1) we see that \( I \) satisfies (*)

- (ii) Let \( I_\omega(t) \) be the steady state response for
  \[
  I''_\omega + 0.1I'_\omega + \frac{1}{C}I_\omega = \sin(\omega t).\n  \]
  Then, by linearity, the steady-state response for (*) is
  \[
  I(t) = I_1(t) + 4I_5(t).
  \]
  Notice that the solution to the homogeneous problem, (i.e. the transient response) will die out as \( t \to \infty \) since the circuit has resistance. Thus, as \( t \to \infty \), we will only observe the steady-state response. Now we calculate the particular solution for (2). Consider
  \[
  \tilde{I}_\omega'' + 0.1\tilde{I}_\omega' + \frac{1}{C}\tilde{I}_\omega = e^{i\omega t}.
  \]
  Let \( \tilde{I}_\omega = Ae^{i\omega t} \). Then \( I_\omega = \text{Im}(\tilde{I}_\omega) \). Substituting, we can determine \( A \) as
  \[
  A \left[ \left( -\omega^2 + \frac{1}{C} \right) + 0.1i\omega \right] = 1, \quad \Rightarrow \quad A = \frac{\left( -\omega^2 + C^{-1} \right) - 0.1i\omega}{(\omega^2 - C^{-1})^2 + .01\omega^2}.
  \]
Taking the imaginary part, we get after a little algebra

\[ I_\omega = \frac{\sin(\omega t + \phi(\omega))}{\triangle(\omega)}, \quad \triangle(\omega) \equiv \left( (\omega^2 - C^{-1})^2 + 0.01\omega^2 \right)^{1/2}, \quad \tan(\phi(\omega)) = \frac{0.1\omega}{(\omega^2 - C^{-1})}, \]

where \( \cos(\phi) < 0 \). Hence, from (3), the steady-state response is

\[ I(t) = \frac{\sin(t + \phi(1))}{\triangle(1)} + \frac{4\sin(5t + \phi(5))}{\triangle(5)}. \quad (4) \]

- (iii) From the formula above the response which oscillates with frequency 1 will be amplified when \( C = 1 \). We obtain a very large amplification of this term since the damping coefficient given by the resistance is very small. For this value of \( C \), the response which oscillates at a frequency of 5 will have a very small amplitude and hence will not be observed in a significant way. Alternatively, the term which oscillates with frequency 5 will be amplified when \( C \approx 1/25 \). For this value of \( C \), the response which oscillates at a frequency of 1 will have a very small amplitude and hence will not be observed in a significant way. Thus, by varying \( C \) the circuit can be tuned to respond to either of the two frequencies. When \( C = 1/81 \), the circuit does not respond to either of the two frequencies of 1 and 5 since both of these oscillations can be seen from (4) to have a very small amplitude.

- (iv) The plots are shown in Fig. 4 for three values of \( C \).

6. Often bumps like the one depicted below are built into roads to discourage speeding.
The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with speed $V$ is given by:

$$
\begin{align*}
    y(t) &= 0 \quad \text{for } t \leq -L/2V \\
    my'' + ky &= \begin{cases} 
    \cos(\pi V t / L) & \text{for } -L/2V \leq t \leq L/2V \\
    0 & \text{for } t \geq L/2V
    \end{cases}
\end{align*}
$$

(The absence of a damping term indicates that the car’s shock absorbers are broken.) Note that the equations are dependent on time only; as the speed is given as $V$, we can write space $x$ in terms of time $t$: $x = Vt$.

(a) Solve this initial value problem; take $m = k = 1$ and $L = \pi$ for convenience. Thus show that the formula for oscillatory motion after the car has traversed the speed bump is $y(t) = A \sin(t)$, where $A$ depends on the speed $V$.

(b) Use Matlab/Octave (or similar) to plot the amplitude $|A|$ of the solution $y(t)$ in part (a) versus the car’s speed $V$. From the graph, estimate the speed that produces the most violent shaking of the vehicle.

Solution:
(a) The solution of this problem can be broken up into 3 parts according to the time $t$:

$$
    y(t) = \begin{cases} 
        y_1(t) & t \leq -\pi/2V \\
        y_2(t) & -\pi/2V \leq t \leq \pi/2V \\
        y_3(t) & t \geq \pi/2V
    \end{cases}
$$

where

$$
    \begin{align*}
        y_1(t) &= 0, \\
        \begin{cases} 
            y_2'' + y_2 = \cos(Vt) \\
            y_2(-\pi/2V) = y_1(-\pi/2V) = 0 \\
            y_2'(-\pi/2V) = y_1'(-\pi/2V) = 0
        \end{cases}
    \end{align*}
\quad \text{and} \quad
\begin{align*}
    y_3'' + y_3 &= 0 \\
    y_3(\pi/2V) &= y_2(\pi/2V) = 0 \\
    y_3'(\pi/2V) &= y_2'(\pi/2V)
\end{align*}
$$

In the above we have taken $m = k = 1$ and $L = \pi$. Note that the initial condition in each initial value problem depends on the previous solution. Since $y_1(t)$ is given to be zero, and the problem asks about the car’s motion after it has traversed the speed bump (for $t \geq \pi/2V$, that is, the problem is to find the solution of the IVP for $y_3$), to solve this problem we have to:

(i) solve the IVP for $y_2$.
(ii) use $y_2$ to generate initial conditions for the IVP for $y_3$.
(iii) Solve the IVP for $y_3$.  

(i) We want to solve
\[
\begin{cases}
y''_2 + y_2 = \cos(Vt) \\
y_2(-\pi/2V) = 0 \\
y_2(-\pi/2V) = 0
\end{cases}
\]
First we find the solution to the homogeneous equation \((y_2)_h'' + (y_2)_h = 0\). Its characteristic equation is
\[r^2 = 1\]
which has roots \(r = \pm i\), so the homogeneous solution is \((y_2)_h = C_1 \cos(t) + C_2 \sin(t)\). To find the particular solution to the inhomogeneous equation \((y_2)_p'' + (y_2)_p = \cos(Vt)\) pose the guess \((y_2)_p(x) = A \cos(Vt) + B \sin(Vt)\). Substituting the guess into the equation, noting that \((y_2)_p''(x) = -V^2 A \cos(Vt) - V^2 B \sin(Vt)\), we obtain \(A(1-V^2) \cos(Vt) + B(1-V^2) \sin(Vt) = \cos(Vt)\), and find that \(A = 1/(1-V^2)\) and \(B = 0\). The particular solution is \((y_2)_p(x) = \cos(Vt)/(1-V^2)\). The general solution is then \(y_2(t) = (y_2)_h(t) + (y_2)_p(t) \Rightarrow y_2(t) = C_1 \cos(t) + C_2 \sin(t) + \cos(Vt)/(1-V^2)\). Applying the initial conditions we find \(C_1 = V \sin(\pi/2V)/(V^2-1)\), \(C_2 = V \cos(\pi/2V)/(V^2-1)\), and the solution to the initial value problem:
\[
y_2(y) = \frac{V}{V^2 - 1} \left( \sin\left(\frac{\pi}{2V}\right) \cos(t) + \cos\left(\frac{\pi}{2V}\right) \sin(t) \right) + \frac{1}{1-V^2} \cos(Vt).
\]
Using the trig identity \(\sin(A + B) = \sin(A) \cos(B) + \sin(B) \cos(A)\), this can be re-written as:
\[
y_2(t) = \frac{1}{V^2 - 1} (V \sin(t + \frac{\pi}{2V}) - \cos(Vt)).
\]

(ii) Now we can generate the initial conditions for the IVP for \(y_3\):
\[
y_3\left(\frac{\pi}{2V}\right) = y_2\left(\frac{\pi}{2V}\right) = \frac{V}{V^2 - 1} \sin\left(\frac{\pi}{V}\right), \text{ and}
\]
\[
y_3'\left(\frac{\pi}{2V}\right) = y_2'\left(\frac{\pi}{2V}\right) = \frac{V}{V^2 - 1} \left( \cos(t + \frac{\pi}{2V}) + \sin(Vt) \right)|_{t=\pi/2V} = \frac{V}{V^2 - 1} (1 + \cos(\frac{\pi}{V}))
\]

(iii) Finally we solve the IVP for \(y_3\). We have \(y_3'' + y_3 = 0\); the solution to this homogeneous equation is \(y_3(t) = C_3 \cos(t) + C_4 \sin(t)\). like in (i). Applying the initial conditions we find:
\[
C_3 = 0 \text{ and } C_4 = \frac{2V}{V^2 - 1} \cos\left(\frac{\pi}{2V}\right).
\]
This step requires some algebraic manipulation and, depending on how you calculate \(C_3\) and \(C_4\), you might have to make use of the trig identities \(\sin(2a) = 2 \sin(a) \cos(a)\), or \(\cos(2a) = 2 \cos^2(a) - 1\), \(\cos(a) \cos(b) + \sin(a) \sin(b) = \cos(a-b)\), or \(\sin(a) \cos(b) - \sin(b) \cos(a) = \sin(a-b)\).

Thus we have found the formula for oscillatory motion after the car has traversed the speed bump is
\[
y_3 = \frac{2V}{V^2 - 1} \cos\left(\frac{\pi}{2V}\right) \sin(t),
\]
which is of the form \(y(t) = A \sin(t)\) where \(A = 2V \cos(\pi/2V)/(V^2-1)\) depends on the speed \(V\).
(b) We’re asked to plot the amplitude $|A|$ and estimate from the plot the speed $V$ that produces the most violent shaking of the vehicle. In (a) we found

$$A = 2V \cos(\pi/2V)/(V^2 - 1).$$

Below we see $|A|$ plotted with respect to the speed $V$.

From the plot we see the maximum amplitude of oscillation $|A|$ occurs when the speed $V$ is around 0.75; that, therefore, is the approximate speed that produces the most violent shaking of the vehicle. Note that you can find this maximum amplitude solving $dA/dV = 0$ for $V$ and verifying that your answer is a local minimum. But then you would be required to solve a transcendental equation!