1. Compute $e^{tA}$ for the following matrices:

(a) $A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(b) $A_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

**Solution: Problem 1:**

(a) In order to exponentiate a matrix, we must first diagonalize the matrix. That is we find a matrix $D$ such that the diagonal entries are the eigenvalues of $A_1$ and a matrix $S$, such that $A_1 = SDS^{-1}$. The eigenvalues of $A_1$ are $\lambda_{1,2} = 1 \pm i$ with corresponding eigenvectors $v_1 = (-i, 1)$ and $v_2 = (i, 1)$. Therefore, we obtain

$$D = \begin{pmatrix} 1 - i & 0 \\ 0 & 1 + i \end{pmatrix}, \quad S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

Finding $S^{-1}$, yields $S^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$.

We now have,

$$e^{Dt} = \begin{pmatrix} e^{(1-i)t} & 0 \\ 0 & e^{(1+i)t} \end{pmatrix}.$$  

It follows that,

$$e^{A_1t} = S e^{Dt} S^{-1} = \begin{pmatrix} e^{(1-i)t} + e^{(1+i)t} \\ e^{(1-i)t} - e^{(1+i)t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{(1-i)t} \\ e^{(1+i)t} \end{pmatrix}$$

Observe that $e^{(1 \pm i)t} = e^t e^{\pm it}$ and recall Euler’s formula, $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$. Applying these we obtain,

$$e^{A_1t} = e^{t} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$  

(b) We see the characteristic polynomial is $(1 - \lambda)^2 = 0$. Thus, $\lambda = 1$ is the only eigenvalue and has corresponding eigenvector $v_1 = (0, 1)$. Hence, we see that we have a defective matrix and cannot diagonalize as we did in part (a). Therefore, let’s view this eigenvalue/eigenvector pair as a solution to $x' = A_2x$. Hence, $x_1 = e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The ansatz for our second solutions is $x_2 = te^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{t}d$. Plug this into $x' = A_2x$ to get

$$te^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{t}d = te^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{t}Ad.$$
Notice that the first terms of the LHS and the RHS are identical and cancel. After canceling $e^t$ from each term, we obtain the equation

\[
\begin{pmatrix} d_1 \\ 1 + d_2 \end{pmatrix} = A \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ -3d_1 + d_2 \end{pmatrix}.
\]

This leads to the condition $d_1 = 1/3$ and $d_2$ is arbitrary. Thus, take $d_2 = 0$. We now obtain, $x_2 = te^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$. We can now form

\[
X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \begin{pmatrix} -1/3 \\ te^t \end{pmatrix} \end{pmatrix}.
\]

It follows that,

\[
e^tA_2 = X(t)X^{-1}(0) = \begin{pmatrix} e^t \\ -3te^t \\ e^t \end{pmatrix}.
\]

2. The functions $\sinh(t)$ and $\cosh(t)$ are defined by

\[
\sinh(t) = \frac{1}{2} (e^t - e^{-t}) , \quad \cosh(t) = \frac{1}{2} (e^t + e^{-t}) .
\]

You should check that you agree that $\sinh(0) = 0$, $\cosh(0) = 1$, $\cosh(t) = \cosh(-t)$, $\sinh(t) = -\sinh(-t)$ and

\[
\frac{d}{dt} \sinh(t) = \cosh(t) , \quad \frac{d}{dt} \cosh(t) = \sinh(t) .
\]

- (a) Verify (by plugging in) that $y = e^{-kt}$, $y = e^{kt}$ and $y = A \sinh(k(t - t_0)) + B \cosh(k(t - t_0))$ are solutions of $y'' - k^2 y = 0$ for all $t > 0$ and for all constants $k > 0$, $t_0$, $A$ and $B$.

- (b) Using sinh and cosh, find the solution $y = y(t)$ to the following initial value problems:

\[
y'' - 16y = 0 , \quad y(0) = 1 , \quad y'(0) = 0 \\
y'' - 9y = 0 , \quad y(3) = 1 , \quad y'(3) = 0
\]

For the second problem it is more convenient to look for a solution in the form $y = A \sinh(k(t - t_0)) + B \cosh(k(t - t_0))$ where $k = 3$ and $t_0 = 3$.

Solution:

- (a) Let $y = e^{\pm kt}$, then $y' = \pm ke^{\pm kt}$ and $y'' = k^2 e^{\pm kt}$. Hence $y'' - k^2 y = 0$ is satisfied. Now since $\sinh(k(t - t_0))$ and $\cosh(k(t - t_0))$ are both linear combinations of $e^{kt}$ and $e^{-kt}$, it follows by the superposition principle that they are both solutions.
• (b) The solution to \( y'' - 16y = 0 \) can be written as \( y = c_1 \sinh(4t) + c_2 \cosh(4t) \). Now, from the properties of \( \sinh(t) \) and \( \cosh(t) \) given above, we get \( c_2 = 1 \) and \( c_1 = 0 \). Thus, the solution is \( y = \cosh(4t) = \frac{1}{2} \left( e^{4t} + e^{-4t} \right) \). The solution to \( y'' - 9y = 0 \) can be written as \( y = c_1 \sinh(3(t - 3)) + c_2 \cosh(3(t - 3)) \). Now, from the properties of \( \sinh(t) \) and \( \cosh(t) \) given above, we get \( c_2 = 1 \) and \( c_1 = 0 \). Thus, the solution is \( y = \cosh(3(t - 3)) = \frac{1}{2} \left( e^{3(t-3)} + e^{-3(t-3)} \right) \).

3. Solve \( y'' + 25y = \cos(5.6t) \) with \( y(0) = 0 \) and \( y'(0) = 0 \). Use Matlab/Octave (or similar) to make an accurate plot of the solution. What do you observe?

**Solution:** For the homogeneous problem let \( y = e^{rt} \). Then, \( r^2 + 25 = 0 \) so that \( r = \pm 5i \). The solution to the homogeneous problem is \( y = \sin(5t) \) or \( y = \cos(5t) \). Now calculate the particular solution. Let \( \tilde{y} \) solve \( \tilde{y}'' + 25\tilde{y} = e^{i\omega t} \), \( \omega \equiv 5.6 \).

Then \( y = \text{Re}(\tilde{y}) \). Let \( \tilde{y} = Ae^{i\omega t} \). Substitute to get \( A = \frac{\text{Re}(\omega)}{25 - \omega^2} \).

The general solution is
\[
y = c_1 \cos(5t) + c_2 \sin(5t) + \frac{\text{Re}(\omega)}{25 - \omega^2}.
\]

Satisfying the initial conditions \( y(0) = y'(0) = 0 \), we get \( c_2 = 0 \), \( c_1 = -1/(25 - \omega^2) \). Thus, using a trig identity we can write
\[
y = \frac{1}{(25 - \omega^2)} \left[ \cos(\omega t) - \cos(5t) \right] = \frac{1}{3.18} \sin(0.3t) \sin(5.3t).
\]

The solution, which exhibits the phenomenon of beats as described in class, is plotted below in Fig. 2.

4. (This is an Important Problem:) Consider \( y'' + py' + y = F_0 \sin(\omega t) \), where \( p > 0 \) and \( F_0 \) are constants. Use Matlab/Octave (or similar) to plot the amplitude of the steady-state (long-term) response as a function of \( \omega \) on the interval \( \omega > 0 \). Make these plots for several representative values of \( p \). Where is the amplitude a maximum as a function of \( \omega \)?

**Solution:** Let \( \tilde{y} \) satisfy
\[
\tilde{y}'' + p\tilde{y}' + \tilde{y} = F_0 e^{i\omega t}.
\]

Then \( y_p = \text{Im}(\tilde{y}) \). Substitute \( \tilde{y} = Ae^{i\omega t} \) into (1) to get
\[
A \left[ (1 - \omega^2) + i\omega \right] = F_0,
\]

Thus,
\[
y_p = \text{Im} \left[ \frac{F_0 \left( (1 - \omega^2) - i\omega \right)}{(1 - \omega^2)^2 + p^2\omega^2} (\cos(\omega t) + i \sin(\omega t)) \right].
\]

A simple calculation then yields
\[
y_p = R(\omega) \cos(\omega t - \phi(\omega)),
\]

3
where $R(\omega)$ and $\phi(\omega)$ satisfy

$$R(\omega) = \frac{F_0}{[(1 - \omega^2)^2 + p^2\omega^2]^{1/2}}, \quad \tan(\phi) = \frac{-(1 - \omega^2)}{p\omega}, \quad (*)$$

with $\cos(\phi) < 0$. The amplitude of the steady-state response is $R(\omega)$. It has a maximum, on the interval $\omega \geq 0$, when the denominator in $(*)$ has a minimum on this interval. Using a little calculus, we find that this occurs at $\omega = \omega_c$, where

$$\omega_c = \sqrt{1 - \frac{p^2}{2}}, \quad 0 < p < \sqrt{2}; \quad \omega_c = 0, \quad p \geq \sqrt{2}.$$ 

Thus, if the damping coefficient $p$ is not too large (i.e. $p < \sqrt{2}$), then the system has its largest response at a frequency $\omega_c \neq 0$. The maximum amplitude $R(\omega_c)$ increases as $p$ decreases and becomes unbounded as $p \to 0$. The frequency $\omega_c$ tends to the resonant frequency of 1 as $p \to 0$. When $F_0 = 1$, we plot $R(\omega)$ versus $\omega$ below in Fig. 3 for $p \approx 0$, for $p = 1, p = \sqrt{2}$ and $p > \sqrt{2}$.

5. (Tuning a circuit) Consider an $RLC$ circuit in series with an A.C. voltage source given by $V(t) = -\cos t - 4/5\cos(5t)$. Suppose that $R = .1$ Ohms, $L = 1$ Henry but that we are capable of adjusting the capacitance $C$ of the capacitor.

- (i) Show that the current in the circuit satisfies

$$I'' + 0.1I' + \frac{1}{C}I = \sin t + 4\sin(5t)$$

- (ii) Calculate the steady-state (long-term) solution. This is the solution after any transient terms have decayed to zero.
Figure 2: Plot of $R(\omega)$ in (*) versus $\omega$ for $p = 0.2$ (heavy solid curve), $p = 1.0$ (solid curve), $p = \sqrt{2}$ (dotted curve), and $p = 2$ (widely spaced dots).

- (iii) What are the critical values of $C$ for which the steady-state current $I$ will (roughly) be periodic with a frequency of either 1 or 5? Interpret this result in terms of the tuning of a circuit.
- (iv) Use Matlab/Octave (or similar) to plot the steady-state solution when $C = 1$, $C = 1/25$ and $C = 1/81$.

Solution:
- (i) The voltage drops across the three elements is equal to the applied voltage. This yields,

$$LQ'' + RQ + Q/C = -\cos t - \frac{4}{5} \cos(5t),$$

where $Q$ is the charge on the capacitor. Now let $L = 1$ Henry, $R = .1$ Ohm and note that $I = Q'$, where $I$ is the current flowing through the circuit. Thus, upon differentiating (1) we see that $I$ satisfies (*).
- (ii) Let $I_\omega(t)$ be the steady state response for

$$I''_\omega + 0.1I'_\omega + \frac{1}{C}I_\omega = \sin(\omega t).$$

Then, by linearity, the steady-state response for (*) is

$$I(t) = I_1(t) + 4I_5(t).$$

Notice that the solution to the homogeneous problem, (i.e. the transient response) will die out as $t \to \infty$ since the circuit has resistance. Thus, as $t \to \infty$, we will only observe the steady-state response. Now we calculate the particular solution for (2). Consider

$$\tilde{I}''_\omega + 0.1\tilde{I}'_\omega + \frac{1}{C}\tilde{I}_\omega = e^{i\omega t}.$$
Let $\tilde{I}_\omega = A e^{i\omega t}$. Then $I_\omega = \text{Im}(\tilde{I}_\omega)$. Substituting, we can determine $A$ as

$$A \left[ \left( -\omega^2 + \frac{1}{C} \right) + 0.1i\omega \right] = 1, \quad \Rightarrow \quad A = \frac{(-\omega^2 + C^{-1}) - 0.1i\omega}{(\omega^2 - C^{-1})^2 + 0.01\omega^2}.$$  

Taking the imaginary part, we get after a little algebra

$$I_\omega = \frac{\sin(\omega t + \phi(\omega))}{\Delta(\omega)}, \quad \Delta(\omega) \equiv \left[ (\omega^2 - C^{-1})^2 + 0.01\omega^2 \right]^{1/2}, \quad \tan(\phi(\omega)) \equiv \frac{0.1\omega}{(\omega^2 - C^{-1})},$$

where $\cos(\phi) < 0$. Hence, from (3), the steady-state response is

$$I(t) = \frac{\sin(t + \phi(1))}{\Delta(1)} + 4\frac{\sin(5t + \phi(5))}{\Delta(5)}. \quad (4)$$

• (iii) From the formula above the response which oscillates with frequency 1 will be amplified when $C = 1$. We obtain a very large amplification of this term since the damping coefficient given by the resistance is very small. For this value of $C$, the response which oscillates at a frequency of 5 will have a very small amplitude and hence will not be observed in a significant way. Alternatively, the term which oscillates with frequency 5 will be amplified when $C \approx 1/25$. For this value of $C$, the response which oscillates at a frequency of 1 will have a very small amplitude and hence will not be observed in a significant way. Thus, by varying $C$ the circuit can be tuned to respond to either of the two frequencies. When $C = 1/81$, the circuit does not respond to either of the two frequencies of 1 and 5 since both of these oscillations can be seen from (4) to have a very small amplitude.

• (iv) The plots are shown in Fig. 4 for three values of $C$.

6. Often bumps like the one depicted below are built into roads to discourage speeding.
The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with speed $V$ is given by:

$$y(t) = 0 \text{ for } t \leq -\frac{L}{2}V$$

$$m\ddot{y} + ky = \begin{cases} \cos(\pi V t / L) & \text{for } -\frac{L}{2}V \leq t \leq \frac{L}{2}V \\ 0 & \text{for } t \geq \frac{L}{2}V \end{cases}.$$  

(The absence of a damping term indicates that the car’s shock absorbers are broken.) Note that the equations are dependent on time only; as the speed is given as $V$, we can write space $x$ in terms of time $t$: $x = Vt$.

(a) Solve this initial value problem; take $m = k = 1$ and $L = \pi$ for convenience. Thus show that the formula for oscillatory motion after the car has traversed the speed bump is $y(t) = A\sin(t)$, where $A$ depends on the speed $V$.

(b) Use Matlab/Octave (or similar) to plot the amplitude $|A|$ of the solution $y(t)$ in part (a) versus the car’s speed $V$. From the graph, estimate the speed that produces the most violent shaking of the vehicle.

**Solution:**

(a) The solution of this problem can be broken up into 3 parts according to the time $t$:

$$y(t) = \begin{cases} y_1(t) & t \leq -\frac{\pi}{2V} \\ y_2(t) & -\frac{\pi}{2V} \leq t \leq \frac{\pi}{2V} \\ y_3(t) & \geq \frac{\pi}{2V} \end{cases},$$

where

$$y_1(t) = 0, \quad \begin{cases} y_2'' + y_2 = \cos(\pi V t) \\ y_2(-\pi / 2V) = y_1(-\pi / 2V) = 0 \\ y_2'(-\pi / 2V) = y_1'(-\pi / 2V) = 0 \end{cases} \quad \text{and} \quad \begin{cases} y_3'' + y_3 = 0 \\ y_3(\pi / 2V) = y_2(\pi / 2V) = 0 \\ y_3'(-\pi / 2V) = y_2'(-\pi / 2V) \end{cases} \quad \text{for } t \geq -\frac{\pi}{2V}.$$  

In the above we have taken $m = k = 1$ and $L = \pi$. Note that the initial condition in each initial value problem depends on the previous solution. Since $y_1(t)$ is given to be zero, and the problem asks about the car’s motion after it has traversed the speed bump (for $t \geq \frac{\pi}{2V}$, that is, the problem is to find the solution of the IVP for $y_3$), to solve this problem we have to:

(i) solve the IVP for $y_2$.

(ii) use $y_2$ to generate initial conditions for the IVP for $y_3$.

(iii) Solve the IVP for $y_3$. 

We want to solve
\[
\begin{align*}
\begin{cases}
y''_2 + y_2 = \cos(Vt) \\
y_2(-\pi/2V) = 0 \\
y_2(-\pi/2V) = 0
\end{cases}
\end{align*}
\]
First we find the solution to the homogeneous equation \((y_2)_h'' + (y_2)_h = 0\). Its characteristic equation is
\(r^2 = 1\) which has roots \(r = \pm i\), so the homogeneous solution is \((y_2)_h = C_1 \cos(t) + C_2 \sin(t)\). To find the particular solution to the inhomogeneous equation \((y_2)_p'' + (y_2)_p = \cos(Vt)\) pose the guess \((y_2)_p(x) = A \cos(Vt) + B \sin(Vt)\). Substituting the guess into the equation, noting that \((y_2)_p''(x) = -V^2 A \cos(Vt) - V^2 B \sin(Vt)\), we obtain \(A(1-V^2) \cos(Vt) + B(1-V^2) \sin(Vt) = \cos(Vt)\), and find that \(A = 1/(1-V^2)\) and \(B = 0\). The particular solution is \((y_2)_p(x) = \cos(Vt)/(1-V^2)\). The general solution is then \(y_2(t) = (y_2)_h(t) + (y_2)_p(t) \Rightarrow y_2(t) = C_1 \cos(t) + C_2 \sin(t) + \cos(Vt)/(1-V^2)\). Applying the initial conditions we find \(C_1 = V \sin(\pi/2V)/(V^2-1)\), \(C_2 = V \cos(\pi/2V)/(V^2-1)\), and the solution to the initial value problem:
\[
y_2(t) = \frac{V}{V^2-1} \left(\sin\left(\frac{\pi}{2V}\right) \cos(t) + \cos\left(\frac{\pi}{2V}\right) \sin(t)\right) + \frac{1}{1-V^2} \cos(Vt).
\]
Using the trig identity \(\sin(A + B) = \sin(A) \cos(B) + \sin(B) \cos(A)\), this can be re-written as:
\[
y_2(t) = \frac{1}{V^2-1} \left(V \sin(t + \frac{\pi}{2V}) - \cos(Vt)\right).
\]

Now we can generate the initial conditions for the IVP for \(y_3\):
\[
y_3\left(\frac{\pi}{2V}\right) = y_2\left(\frac{\pi}{2V}\right) = \frac{V}{V^2-1} \sin\left(\frac{\pi}{V}\right), \text{ and}
\]
\[
y'_3\left(\frac{\pi}{2V}\right) = y'_2\left(\frac{\pi}{2V}\right) = \frac{V}{V^2-1} \left(\cos(t + \frac{\pi}{2V}) + \sin(Vt)\right)|_{t=\pi/2V} = \frac{V}{V^2-1} \left(1 + \cos\left(\frac{\pi}{V}\right)\right)
\]

Finally we solve the IVP for \(y_3\). We have \(y'''_3 + y_3 = 0\); the solution to this homogeneous equation is \(y_3(t) = C_3 \cos(t) + C_4 \sin(t)\). like in (i). Applying the initial conditions we find:
\[
C_3 = 0 \text{ and } C_4 = \frac{2V}{V^2-1} \cos\left(\frac{\pi}{2V}\right).
\]
This step requires some algebraic manipulation and, depending on how you calculate \(C_3\) and \(C_4\), you might have to make use of the trig identities \(\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)\), or \(\cos(2\alpha) = 2 \cos^2(\alpha) - 1\), \(\cos(a) \cos(b) + \sin(a) \sin(b) = \cos(a-b)\), or \(\sin(a) \cos(b) - \sin(b) \cos(a) = \sin(a-b)\). Thus we have found the formula for oscillatory motion after the car has traversed the speed bump is
\[
y_3 = \frac{2V}{V^2-1} \cos\left(\frac{\pi}{2V}\right) \sin(t),
\]
which is of the form \(y(t) = A \sin(t)\) where \(A = 2V \cos(\pi/2V)/(V^2-1)\) depends on the speed \(V\).
(b) We’re asked to plot the amplitude $|A|$ and estimate from the plot the speed $V$ that produces the most violent shaking of the vehicle. In (a) we found

$$A = 2V \cos(\pi/2V)/(V^2 - 1).$$

Below we see $|A|$ plotted with respect to the speed $V$.

From the plot we see the maximum amplitude of oscillation $|A|$ occurs when the speed $V$ is around 0.75; that, therefore, is the approximate speed that produces the most violent shaking of the vehicle. Note that you can find this maximum amplitude solving $dA/dV = 0$ for $V$ and verifying that your answer is a local minimum. But then you would be required to solve a transcendental equation!

7. Connected tank problem. Consider two drinking water tanks interconnected with each other, as shown in the figure below. There is a flow rate of $2Q$ from tank 1 to tank 2 and a flow rate of $Q$ from tank 2 to tank 1. Each tank receives an inflow of $Q$ and water is taken from tank 2 at a discharge rate of $2Q$. Assume due to some incident the tanks contain some masses of salt denoted by $M_1(t)$ and $M_2(t)$. (Perhaps there was an accidental contamination or salt is present in one or both of the inflows).

Let $V_0$ be the volume of each tank and $C_1(t) = M_1(t)/V_0$ and $C_2(t) = M_2(t)/V_0$ be the concentrations of salt in tanks 1 and 2, respectively. Assume that both tanks are well mixed at all times. Writing the conservation of mass in each tanks results in a $2 \times 2$ linear system:

$$\frac{d}{dt} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \frac{Q}{V_0} \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} + \frac{Q}{V_0} \begin{pmatrix} C_{1i}(t) \\ C_{2i}(t) \end{pmatrix}$$

(1)
Assume $Q = 1000 \text{ L/hr}$ and $V_0 = 100 \text{ m}^3$.

(a) (Washout problem) Suppose that the inflows are both clean and contain no salt. Find the the matrix exponential for this system. Use this matrix exponential to find the solutions to the following problems:

\[
\begin{pmatrix}
C_1(0) \\
C_2(0)
\end{pmatrix} =
\begin{pmatrix}
100 \\
0
\end{pmatrix} \text{ mg/L},
\begin{pmatrix}
C_1(0) \\
C_2(0)
\end{pmatrix} =
\begin{pmatrix}
0 \\
100
\end{pmatrix} \text{ mg/L},
\begin{pmatrix}
C_1(0) \\
C_2(0)
\end{pmatrix} =
\begin{pmatrix}
100 \\
100
\end{pmatrix} \text{ mg/L}.
\]

Plot the time variation of $C_1$ and $C_2$ for each case. Label axes and indicate units.

(b) (Transient contamination of inflow) Consider an incident in which the inflow to tank 1 contains a constant salt concentration of 100 mg/L over a 30 hour period as described in the figure below, while $C_{in}^2$ remains zero. Solve the new initial value problem with this forcing and the initial conditions:

\[
\begin{pmatrix}
C_1(0) \\
C_2(0)
\end{pmatrix} =
\begin{pmatrix}
100 \\
0
\end{pmatrix} \text{ mg/L},
\]
Plot the time variation of $C_1$ and $C_2$ on the same figure. You can do this by solving analytically and plotting, or by solving numerically (ode45) and plotting. Label axes and indicate units.

**Hint:** You may want to use the attached Matlab code. Modify this code to take into account three time intervals of different forcing vectors. Note that each interval has its own initial conditions.

**Solution: Problem 5:**
(a) Since both inflows are clean and contain no salt, there is no forcing function for the first part. We begin by simplifying $Q/V_0$. Observe that, $V_0 = 100m^3 = 100000L$. Thus, $Q/V_0 = 1000/100000 = .01hr^{-1}$. Substituting this into (3) yields,

$$\frac{d}{dt} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \begin{pmatrix} -0.02 & .01 \\ .02 & -0.03 \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}.$$  \hspace{1cm} (2)

We now proceed by finding the matrix exponential of the matrix. The matrix has eigenvalues, $\lambda_1 = -0.01$ and $\lambda_2 = -0.04$. The corresponding eigenvectors are $v_1 = (-1, 2)$ and $v_2 = (1, 1)$. Therefore, the matrix is diagonalizable with,

$$D = \begin{pmatrix} -0.04 & 0 \\ 0 & -0.01 \end{pmatrix}, S = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, S^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$$

so that $A = SDS^{-1}$. Therefore,

$$X(t) = e^{At} = Se^{Dt}S^{-1} = \frac{1}{3} \begin{pmatrix} 2e^{-0.01t} + e^{-0.04t} & e^{-0.01t} - e^{-0.04t} \\ 2e^{-0.01t} - 2e^{-0.04t} & e^{-0.01t} + 2e^{-0.04t} \end{pmatrix}. $$

For each initial condition we compute, $C(t) = X(t)C(0)$:

$$C(t) = \frac{1}{3} \begin{pmatrix} 2e^{-0.01t} + e^{-0.04t} & e^{-0.01t} - e^{-0.04t} \\ 2e^{-0.01t} - 2e^{-0.04t} & e^{-0.01t} + 2e^{-0.04t} \end{pmatrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix} = \frac{100}{3} \begin{pmatrix} 2e^{-0.01t} + e^{-0.04t} \\ 2e^{-0.01t} - 2e^{-0.04t} \end{pmatrix}$$

$$C(t) = \frac{1}{3} \begin{pmatrix} 2e^{-0.01t} + e^{-0.04t} & e^{-0.01t} - e^{-0.04t} \\ 2e^{-0.01t} - 2e^{-0.04t} & e^{-0.01t} + 2e^{-0.04t} \end{pmatrix} \begin{pmatrix} 0 \\ 100 \end{pmatrix} = \frac{100}{3} \begin{pmatrix} e^{-0.01t} + e^{-0.04t} \\ e^{-0.01t} + 2e^{-0.04t} \end{pmatrix}$$
\( C(t) = \frac{1}{3} \left( \frac{2e^{-0.04t} + e^{-0.04t}}{2e^{-0.01t} - 2e^{-0.04t}} \right) \left( \frac{2e^{-0.01t} + e^{-0.01t} - e^{-0.04t}}{e^{-0.01t} + 2e^{-0.04t}} \right) \left( \frac{100}{100} \right) = \frac{100}{3} \left( \frac{2e^{-0.01t} + e^{-0.04t}}{2e^{-0.01t} - 2e^{-0.04t}} \right) \left( \frac{e^{-0.01t} - e^{-0.04t}}{e^{-0.01t} + 2e^{-0.04t}} \right) = 100 \left( e^{-0.01t} \right) \)

(b) See Figure 4 and accompanying Matlab code.

Matlab/Octave code

```
*************** Code for Problem 2 ***************
```

[X1, X2] = meshgrid(0:.1:2,0:.1:2);
F1=@(x1,x2) x1-x1.^2-x1.*x2;
F2=@(x1,x2) .5*x2-.25*x2.^2-.75*x1.*x2;

% make matrices U1 and U2 containing the vector components
U1 = F1(X1,X2);
U2 = F2(X1,X2);

% rescale arrow length
L=0.05;
len=length(U1);
for(i=1:len)
    for(j=1:len)
        nr=sqrt(U1(i,j)^2+U2(i,j)^2);
        U1(i,j)=L*U1(i,j)/nr;
        U2(i,j)=L*U2(i,j)/nr;
    end
end

% plot the vector field
quiver(X1,X2,U1,U2,0)
hold on;

% set the time range to run on
tr=[0 20];

f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))];
for i=0.1:0.3:2
    ip=[2;i]; %set initial point along line x1=2
    [~, xout]=ode45(f,tr,ip);
    plot(xout(:,1),xout(:,2),'r')
    ip=[0.05;i]; %now set initial point along line x1=0.05, etc...
    [~, xout]=ode45(f,tr,ip);
    plot(xout(:,1),xout(:,2),'r')
    ip=[i;0.05];
    [~, xout]=ode45(f,tr,ip);
    plot(xout(:,1),xout(:,2),'r')
    ip=[i;2];
    [~, xout]=ode45(f,tr,ip);
    plot(xout(:,1),xout(:,2),'r')
end;
axis([-0.1 2 -.1 2]);
box on;

%%%%%%%%%%%%%%%%%%%%% Code for Problem 4 %%%%%%%%%%%%%%%%%%%

% set the value of K
K=0.5;

[X1, X2] = meshgrid(0:.1:3,0:.1:3);
\[ F_1(x_1, x_2) = 10 \cdot (1 - \frac{1}{K} x_1) \cdot x_1 - 10 \cdot x_1 \cdot x_2 / (1 + x_1) \]
\[ F_2(x_1, x_2) = -x_2 + 2.5 \cdot x_1 \cdot x_2 / (1 + x_1) \]

% make matrices U1 and U2 containing the vector components
U1 = F1(X1,X2);
U2 = F2(X1,X2);

% rescale arrow length
L=0.05;
len=length(U1);
for (i=1:len)
    for (j=1:len)
        nr=sqrt(U1(i,j)^2+U2(i,j)^2);
        U1(i,j)=L*U1(i,j)/nr;
        U2(i,j)=L*U2(i,j)/nr;
    end
end

% plot the vector field
quiver(X1,X2,U1,U2,0)
hold on;

% set and the time range to run on
tr=[0 10];

f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))];
for i=0.1:0.6:2
    ip=[2.5;i]; %set initial point along line x1=2.5
    [~, xout]=ode45(f, tr, ip);
    plot(xout(:,1),xout(:,2), 'r')
end;

end;

% mark the equilibrium points
plot([0 K 2/3],[0 0 5/3-10/9/K], 'Marker','o', 'MarkerEdgeColor','k', ...
     'Linewidth',2, 'LineStyle','none');
axis([-0.1 2.5 -.1 2]);
box on;

% Problem 5a
% Change IC to match
close all;
clear all;
clc;
A=[-2 1;2 -3]/100;

% setting up the homogeneous system
F1=@(x1,x2) A(1,1)*x1+A(1,2)*x2;
F2=@(x1,x2) A(2,1)*x1+A(2,2)*x2;

% no forcing functions
f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))];

% setting the time range and IC for ode45:
tr=[0:0.1:1000];

% Uncomment the IC you want
ic=[100;0];
% ic=[0;100];
% ic=[100;100];

% now run ode45 and put the output into tout and xout:
[tout xout]=ode45(f,tr,ic);

figure;
plot(tout,xout(:,1),'-o','LineWidth',2.5);hold on;
plot(tout,xout(:,2),'r','LineWidth',2.5);
xlim([tr(1) tr(end)]);
%ylim([0 30]);
set(gca,'Fontsize',20);
xlabel('time (hr)','Fontsize',20); ylabel('C_{1},C_{2}','Fontsize',20);
legend('C_{1}','C_{2}');
%set(gca,'dataaspectratio',[1 1 1])
print -depsc 'two_tank_mixing_c2100.eps';

% Problem 5b

close all;
clear all;
clc;
A=[-2 1;2 -3]/100;

% solution for the interval: t = 0 to 20 hr
% setting up the homogeneous system
F1=@(x1,x2) A(1,1)*x1+A(1,2)*x2;
F2=@(x1,x2) A(2,1)*x1+A(2,2)*x2;
% now adding the forcing functions
f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))];

% setting the time range and IC for ode45:
tr=[0:0:1:20];
ic=[100;0];
% now run ode45 and put the output into tout and xout:
[tout xout]=ode45(f,tr,ic);

%figure;

plot(tout,xout(:,1),'b','LineWidth',2.5);hold on;
plot(tout,xout(:,2),'r','LineWidth',2.5);

%%
% solution for the interval: t = 20 to 50 hr
% setting up the homogeneous system
F1=@(x1,x2) A(1,1)*x1+A(1,2)*x2;
F2=@(x1,x2) A(2,1)*x1+A(2,2)*x2;
% now adding the forcing functions
f=@(t,x) [F1(x(1),x(2))+100/100; F2(x(1),x(2))];

% setting the time range and IC for ode45:
tr=[20:0.1:50];
ic=[xout(end,1);xout(end,2)]; % the IC comes from the solution of the previous interval at the t=20 hr
% now run ode45 and put the output into tout and xout:
[tout xout]=ode45(f,tr,ic);

%figure;

plot(tout,xout(:,1),'b','LineWidth',2.5);hold on;
plot(tout,xout(:,2),'r','LineWidth',2.5);

%%
% solution for the interval: t = 50 hr to infinity
% setting up the homogeneous system
F1=@(x1,x2) A(1,1)*x1+A(1,2)*x2;
F2=@(x1,x2) A(2,1)*x1+A(2,2)*x2;
% now adding the forcing functions
f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))];

% setting the time range and IC for ode45:
tr=[50:0.1:300];
ic=[xout(end,1);xout(end,2)];% the IC comes from the solution of the previous interval at the t=50 hr
% now run ode45 and put the output into tout and xout:
[tout xout]=ode45(f,tr,ic);

%figure;

plot(tout,xout(:,1),'b','LineWidth',2.5);hold on;
plot(tout,xout(:,2),'r','LineWidth',2.5);

set(gca,'Fontsize',20);
xlabel('time (hr)','Fontsize',20); ylabel('C{1},C{2}','Fontsize',20);
legend('C{1}', 'C{2}');
set(gca,'dataaspectratio',[1 1 1])
print -depsc 'two_tank_mixing_incident.eps';