1. Compute $e^{tA}$ for the following matrices:
   (a) 
   $$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
   (b) 
   $$A_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

2. The functions $\sinh(t)$ and $\cosh(t)$ are defined by
   $$\sinh(t) = \frac{1}{2} (e^t - e^{-t}), \quad \cosh(t) = \frac{1}{2} (e^t + e^{-t}).$$

   You should check that you agree that $\sinh(0) = 0$, $\cosh(0) = 1$, $\cosh(t) = \cosh(-t)$, $\sinh(t) = -\sinh(-t)$ and
   $$\frac{d}{dt} \sinh(t) = \cosh(t), \quad \frac{d}{dt} \cosh(t) = \sinh(t).$$

   (a) Verify (by plugging in) that $y = e^{-kt}$, $y = e^{kt}$ and $y = A \sinh(k(t-t_0)) + B \cosh(k(t-t_0))$ are solutions of $y'' - k^2 y = 0$ for all $t > 0$ and for all constants $k > 0$, $t_0$, $A$ and $B$.

   (b) Using sinh and cosh, find the solution $y = y(t)$ to the following initial value problems:
   $$y'' - 16y = 0, \quad y(0) = 1, \quad y'(0) = 0$$
   $$y'' - 9y = 0, \quad y(3) = 1, \quad y'(3) = 0$$

   For the second problem it is more convenient to look for a solution in the form $y = A \sinh(k(t-t_0)) + B \cosh(k(t-t_0))$ where $k = 3$ and $t_0 = 3$.

3. Solve $y'' + 25y = \cos(5.6t)$ with $y(0) = 0$ and $y'(0) = 0$. Use Matlab/Octave (or similar) to make an accurate plot of the solution. What do you observe?

4. (This is an Important Problem:) Consider $y'' + py' + y = F_0 \sin(\omega t)$, where $p > 0$ and $F_0$ are constants. Calculate the steady state response (after any transients have decayed) and then plot using Matlab as a function of $\omega$ on the interval $\omega > 0$. Make these plots for $p = 0.1$, $p = 1$ and $p = 10$. In each case, where is the amplitude a maximum as a function of $\omega$?

5. (Tuning a circuit) Consider an $RLC$ circuit in series with an A.C. voltage source given by $V(t) = -\cos t - 4/5 \cos(5t)$. Suppose that $R = .1$ Ohms, $L = 1$ Henry but that we are capable of adjusting the capacitance $C$ of the capacitor.

   (i) Show that the current in the circuit satisfies
   $$I'' + 0.1I' + \frac{1}{C} I = \sin t + 4 \sin(5t)$$
• (ii) Calculate the steady-state (long-term) solution. This is the solution after any transient terms have decayed to zero.

• (iii) What are the critical values of $C$ for which the steady-state current $I$ will (roughly) be periodic with a frequency of either 1 or 5? Interpret this result in terms of the tuning of a circuit.

• (iv) Use Matlab/Octave (or similar) to plot the steady-state solution when $C = 1$, $C = 1/25$ and $C = 1/81$.

6. Often bumps like the one depicted below are built into roads to discourage speeding.

The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with speed $V$ is given by:

\[ y(t) = 0 \quad \text{for } t \leq -L/2V \]
\[ my'' + ky = \begin{cases} 
\cos(\pi t/L) & \text{for } -L/2V \leq t \leq L/2V \\
0 & \text{for } t \geq L/2V 
\end{cases}. \]

(The absence of a damping term indicates that the car’s shock absorbers are broken.) Note that the equations are dependent on time only; as the speed is given as $V$, we can write space $x$ in terms of time $t$: $x = Vt$.

(a) Solve this initial value problem; take $m = k = 1$ and $L = \pi$ for convenience. Thus show that the formula for oscillatory motion after the car has traversed the speed bump is $y(t) = A\sin(t)$, where $A$ depends on the speed $V$.

(b) Use Matlab/Octave (or similar) to plot the amplitude $|A|$ of the solution $y(t)$ in part (a) versus the car’s speed $V$. From the graph, estimate the speed that produces the most violent shaking of the vehicle.
7. Consider two drinking water tanks interconnected with each other, as shown in the figure below. There is a flow rate of $2Q$ from tank 1 to tank 2 and a flow rate of $Q$ from tank 2 to tank 1. Each tank receives an inflow of $Q$ and water is taken from tank 2 at a discharge rate of $2Q$. Assume due to some incident the tanks contain some masses of salt denoted by $M_1(t)$ and $M_2(t)$. (Perhaps there was an accidental contamination or salt is present in one or both of the inflows).

Let $V_0$ be the volume of each tank and $C_1(t) = M_1(t)/V_0$ and $C_2(t) = M_2(t)/V_0$ be the concentrations of salt in tanks 1 and 2, respectively. Assume that both tanks are well mixed at all times. Writing the conservation of mass in each tanks results in a $2 \times 2$ linear system:

$$
\begin{align*}
\frac{d}{dt} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} &= \frac{Q}{V_0} \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} + \frac{Q}{V_0} \begin{pmatrix} C_{in}^1(t) \\ C_{in}^2(t) \end{pmatrix}
\end{align*}
$$

Assume $Q = 1000 \text{ L/hr}$ and $V_0 = 100 \text{ m}^3$.

(a) (Washout problem) Suppose that the inflows are both clean and contain no salt. Find the matrix exponential for this system. Use this matrix exponential to find the solutions to the following problems:

$$
\begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix} \text{ mg/L,} \quad \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \end{pmatrix} \text{ mg/L,} \quad \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \text{ mg/L.}
$$

Plot the time variation of $C_1$ and $C_2$ for each case. Label axes and indicate units.

(b) (Transient contamination of inflow) Consider an incident in which the inflow to tank 1 contains a constant salt concentration of 100 mg/L over a 30 hour period as described in the figure below, while $C_{in}^2$
remains zero. Solve the new initial value problem with this forcing and the initial conditions:

\[
\begin{pmatrix}
  C_1(0) \\
  C_2(0)
\end{pmatrix} = \begin{pmatrix}
  100 \\
  0
\end{pmatrix} \text{ mg/L},
\]

Plot the time variation of \( C_1 \) and \( C_2 \) on the same figure. You can do this by solving analytically and plotting, or by solving numerically (ode45) and plotting. Label axes and indicate units.

*Hint:* You may want to use the attached Matlab code. Modify this code to take into account three time intervals of different forcing vectors. Note that each interval has its own initial conditions.