1. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}. $$

Also, describe the behavior of the solution for large $t$.

Solution: Problem 1:
The characteristic polynomial is

$$\lambda^2 + 5\lambda + 4 = 0$$

with roots $-1$ and $-4$. The eigenvector equation for $-1$ is $-x + y = 0$, and that for $-4$ is $x + 2y = 0$. The solutions are therefore

$$y = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}. $$

For the initial conditions given, we solve

$$c_1 + 2c_2 = -3$$

$$c_1 - c_2 = 3,$$

with solutions $c_1 = 1$ and $c_2 = -2$. All solutions decay exponentially to 0 as both eigenvalues are negative.

2. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. $$

Solution: Problem 2:
The characteristic polynomial is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

so $\lambda = 2$ is an eigenvalue with multiplicity 2. The eigenvector equation is $-2x + y = 0$, leading to an eigenvector $\xi = (1, 2)$. We solve

$$(A - 2I)\eta = \xi$$

to get an equation $-2x + y = 1$ for $\eta$, or $\eta = (0, 1)$. Therefore, the solutions are

$$y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. $$
For the initial conditions we solve

\[ c_1 = 1 \]
\[ 2c_1 + c_2 = 3, \]

leading to \( c_1 = 1, \) \( c_2 = 1. \) All solutions grow exponentially fast as \( t \to \infty. \)

3. Find the solution \( y(t) \) to

\[ y' = \begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]

Solution: Problem 3:
The characteristic polynomial is

\[ \lambda^2 + 6\lambda + 13 \]

leading to roots \( \lambda_{1,2} = -3 \pm 2i. \) The corresponding eigenvector equation is \(-ix + y = 0.\) Therefore the real solutions are the real and imaginary parts of

\[ e^{(-3+2i)t} \left( \begin{array}{c} 1 \\ i \end{array} \right) = e^{-3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + ie^{-3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}. \]

The solution is therefore,

\[ y = c_1 e^{-3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}. \]

The initial conditions give \( c_1 = 1, c_2 = 2. \) Solutions oscillate and decay.

4. Solve the following system of differential equations with initial conditions and sketch the solution. On your sketch, label the initial point, initial velocity vector and asymptotic direction.

\[ x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \quad x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

Solution: Problem 4:
The characteristic polynomial is

\[ \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \]

which leads to the roots \( \lambda = -3 \) with multiplicity 2. The eigenvector equation is therefore \( 4x - 4y = 0 \) leading to the corresponding eigenvector is \( \xi = (1, 1). \) We solve

\[(A + 3I)\eta = \xi\]

to get an equation \( 4x - 4y = 1 \) for \( \eta, \) or \( \eta = (1/4, 0). \) Therefore, the solutions are

\[ y = c_1 e^{-3t} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + c_2 e^{-3t} \left[ t \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \left( \frac{1}{2} \right) \right]. \]
For the initial conditions we solve
\[ c_1 + \frac{1}{4}c_2 = -1 \]
\[ c_1 = 1, \]
leading to \( c_1 = 1, c_2 = -8 \). All solutions decay exponentially fast as \( t \to \infty \).

5. Show that all solutions of the system
\[ \mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} \]
approach zero as \( t \to \infty \) if and only if \( a + d < 0 \) and \( ad - bc > 0 \).

Solution: Problem 5:

The only way for all solutions of the system to approach zero as \( t \to \infty \) is if the eigenvalues of the system have negative real parts. Thus, we proceed to find conditions for the eigenvalues.

The characteristic polynomial is \( \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \) and therefore the eigenvalues are given by
\[ \lambda_{1,2} = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} = \frac{a + d}{2} \left( 1 \pm \sqrt{1 - 4 \frac{(ad - bc)}{(a + d)^2}} \right) \]

Observe that if \( (a + d) > 0 \) then the Real Part of \( \lambda_{1,2} \) is certainly positive because the square root quantity is either positive or imaginary and in either case, the real part of \( \lambda_{1,2} \) is positive. If \( (a + d) < 0 \) then the Real Part of \( \lambda_{1,2} < 0 \) provided \( 1 - \sqrt{1 - 4 \frac{(ad - bc)}{(a + d)^2}} > 0 \). This is not true if \( ad - bc < 0 \). Therefore, we require that \( a + d < 0 \) and \( ad - bc > 0 \) for both \( \lambda_{1,2} \) to be negative.
6. Match each direction field to a differential equation, or state that it does not match any of the equations. In each case, the point $(0,0)$ is in the centre of the picture.

A. 

\[
\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \]

B. 

\[
\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \]

C. 

\[
\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \]

D. 

\[
\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} \]

E. 

\[
\begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} \]

Solution: Problem 6:

The eigenvalues of the matrices determine how the pictures look. We have:

1. $\lambda_1 = 1, \lambda_2 = 3$. Two real positive values. Origin is an unstable node. **D**.
2. $\lambda_{1,2} = 2 \pm i$. Complex eigenvalue, positive real part. Origin is an unstable spiral. **A**.
3. $\lambda_{1,2} = \pm \sqrt{2}i$. Pure imaginary eigenvalues. Origin is a center point. **C**.
4. $\lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2}$. Positive and negative eigenvalues. Origin is a saddle. **E**.
5. $\lambda_{1,2} = -3 \pm \sqrt{2}i$. Complex eigenvalues, negative real part. Origin is a stable spiral. **B**.
Part 2

7. Analytically, find the solution $y(t)$ to the initial value problem

$$y' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Solution: Problem 7:

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

which leads to the root $\lambda = 1$ with multiplicity 2. The eigenvector equation is therefore $2y = 0$ leading to the corresponding eigenvector is $\xi = (1, 0)$. We solve

$$(A - I)\eta = \xi$$

to get an equation $2y = 1$ for $\eta$, or $\eta = (1, 1/2)$. Therefore, the solutions are

$$y = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.$$ 

For the initial conditions we solve

$$c_1 + c_2 = 0$$

$$1/2 c_2 = 1,$$

leading to $c_1 = -2$, $c_2 = 2$. All solutions grow exponentially fast as $t \to \infty$.

8. Analytically, find the solution $y(t)$ to

$$y' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ 

Use Matlab to plot the direction field for this system. You can use the uploaded script for this assignment or your own code. Overlay your analytic solution for this particular initial condition on the direction field.

Solution: Problem 8: The characteristic polynomial is

$$\lambda^2 + 5\lambda + 4 = 0$$

with roots $-1, -4$. The eigenvector equation for $-1$ is $-x + y = 0$, and that for $-4$ is $x + 2y = 0$. The solutions are therefore

$$y = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$ 

For the initial conditions given we solve

$$c_1 + 2c_2 = -1$$

$$c_1 - c_2 = 1,$$

with solutions $c_1 = 1/3$, $c_2 = -2/3$. 

5
close all;
clear all;
clc;

% Define the matrix A
A = [-3 2; 1 -2];

% Define the size of window. The window is of size -a<x<a; -a<y<a
a = 2;
[x,y] = meshgrid(-a:a/10:a, -a:a/10:a);
xp = A(1,1)*x + A(1,2)*y;
yp = A(2,1)*x + A(2,2)*y;

% plotting the vector field:
quiver(x,y,xp,yp);hold on;

% define the analytic solution to plot
% Choose t such that x(0) and y(0) match the initial condition
% include enough points for smooth solution
% choose t, end sufficiently long to show qualatative behavior in
% window chosen for the direction field
 t = linspace(0,50,1000);
x = (1/3)*exp(-t) + (-2/3)*exp(-4.*t)*2;
y = (1/3)*exp(-t) + (-2/3)*exp(-4.*t)*(-1);

plot(x,y,'LineWidth', 3.0)
xlabel('y1')
ylabel('y2')
9. Solve the following system of differential equations with initial conditions and sketch the solution. Check that you can do this by hand as best you can (of course you can use Matlab to verify that the key features of your sketch are correct). On your sketch, label the initial point, initial velocity vector and asymptotic direction as \( t \to \infty \).

\[
\begin{align*}
\mathbf{x}' &= \begin{pmatrix} 2 & -2 \\ 4 & 6 \end{pmatrix} \mathbf{x} \\
\mathbf{x}(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\]

Solution: Problem 9:
The characteristic polynomial is 

\[
\lambda^2 - 8\lambda + 20
\]
leading to roots \( \lambda_{1,2} = 4 \pm 2i \). The corresponding eigenvector equation is \((-2 - 2i)x - 2y = 0\). Hence the corresponding eigenvectors are \( \mathbf{v}_{1,2} = \begin{pmatrix} -1 \pm i \\ 2 \end{pmatrix} \). Therefore, the real solutions are the real and imaginary parts of 

\[
e^{(4+2i)t} \begin{pmatrix} -1 + i \\ 2 \end{pmatrix} = e^{4t} \begin{pmatrix} -\cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix} + ie^{4t} \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2\sin(2t) \end{pmatrix}.
\]

The solution is therefore,

\[
\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} -\cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2\sin(2t) \end{pmatrix}.
\]

The initial conditions give \( c_1 = c_2 = \frac{1}{2} \). We can plug in the constants and simplify to obtain

\[
\mathbf{x}(t) = e^{4t} \begin{pmatrix} -\sin(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix}
\]

10. We consider a system of differential equations containing the parameter \( a \).

(a) Determine the eigenvalues in terms of \( a \).
(b) Find the critical value(s) of \( a \) where the qualitative nature of the eigenvalues changes and summarize your findings in words.
(c) Use Matlab to produce a direction field plot for each possible qualitative behaviour of this system and in each case plot several solutions on the direction field. Label the plots with the value of \( a \) that you used and the description of the fixed point at \((0,0)\) (e.g. “Stable node”, “Unstable spiral point”, etc).

Solution: Problem 10:
(a) The characteristic polynomial is 

\[
\lambda^2 - a\lambda + 5 = 0
\]
so the eigenvalues are 

\[
\lambda = \frac{a}{2} \pm \frac{1}{2} \sqrt{a^2 - 20}
\]
(b) The critical values of \( a \) are 0 and \( \pm \sqrt{20} \). The qualitative behaviours are as follow.
Figure 3: Problem 9. Direction field and analytical solution. Students should have a plot by hand.

- $a < -\sqrt{20}$: $\lambda$ are real, negative and distinct, hence the solutions go to zero and the origin is a stable node.
- $a = -\sqrt{20}$: $\lambda$ are real, negative and repeated, hence the solutions go to zero and the origin is a stable node.
- $-\sqrt{20} < a < 0$: $\lambda$ are complex and $Re(\lambda) < 0$, hence the solutions go to zero, spiral inwards and the origin is a stable spiral point.
- $a = 0$: $\lambda = \pm\sqrt{5}i$, hence the solutions are periodic and the origin is the center.
- $0 < a < \sqrt{20}$: $\lambda$ are complex and $Re(\lambda) > 0$, hence the solutions go to infinity, spiral outwards and the origin is an unstable spiral point.
- $a = \sqrt{20}$: $\lambda$ are real, positive and repeated, hence the solutions go to infinity and the origin is an unstable node.
- $a > \sqrt{20}$: $\lambda$ are real, positive and distinct, hence the solutions go to infinity and the origin is an unstable node.

(c) see Figure 4.

11. Give an example of a 2x2 system $\vec{x}'(t) = A\vec{x}$ that has stable equilibria along the line $x_1 = 2x_2$. Use Matlab to plot the direction field with several solutions superimposed to show the behaviour of the system.

**Solution: Problem 11:** Since we need stable equilibria along $x_1 = 2x_2$ it suffices to find a matrix $A$ such that $A\vec{x} = 0$ where $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Many such matrices will do. In its general form, $A = \begin{bmatrix} \alpha & -2\alpha \\ \beta & -2\beta \end{bmatrix}$. A plot and the script used to generate it follow.
Figure 4: Figures for Problem 10. The Matlab commands to produce these figures were similar to that of Problem 8.
Figure 5: Problem 11. Direction field with several solutions superimposed. Various initial points are labelled in the legend.
Problem 11
Direction field and sample solutions (using ode45) for a system that has
stable equilibria along x = 2y.

clear all variables, close all figures
close all;
clear all;
clc;

% define alpha and beta
alpha = 1;
beta = 2;

% form the matrix A as in the solution
A=[alpha -2*alpha;beta -2*beta];

% Define the size of the window we are interested in
a=4;
xp=A(1,1)*x+A(1,2)*y;
yp=A(2,1)*x+A(2,2)*y;

% plotting the vector field:
quiver(x,y,xp,yp);hold on;
%
% using ode45 for several solutions to show the behaviour of the system:

% define function, f=A*x
f=@(t,x) A*x;

% now I will set the initial point
ip=[2;-4];

% and the time range to run on. Lets go from t_initial to t_end:
% Need tsnap to be large enough to capture the qualitative behaviour
% in our window of interest
tr=[0 500];

% now run ode45 and put the output into tout and xout:
[tout xout]=ode45(f,tr,ip);

% plot the phase plane:
plot(xout(:,1),xout(:,2), 'm', 'LineWidth', 3);

% Now run with a few more initial points
ip=[-2;-4];
[tout xout]=ode45(f,tr,ip);
plot(xout(:,1),xout(:,2), 'g', 'LineWidth', 3);

ip=[1;1];
[tout xout]=ode45(f,tr,ip);
plot(xout(:,1),xout(:,2), 'r', 'LineWidth', 3);
12. Find the fundamental matrix and use this to determine the general solution to

\[ \dot{\mathbf{x}}(t) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ -e^t \end{pmatrix} \]

Solution: Problem 12: The characteristic polynomial is

\[ (\lambda - 4)^2 - 1 = 0 \]

so the eigenvalues are \( \lambda_1 = 5 \) and \( \lambda_2 = 3 \). The corresponding eigenvectors are

\[ \xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

Hence the fundamental matrix is

\[ \Phi(t) = \begin{pmatrix} e^{5t} & -e^{3t} \\ e^{5t} & e^{3t} \end{pmatrix}. \]

A particular solution is given by

\[ \mathbf{x}_p(t) = \Phi(t) \int \Phi^{-1}(t) \begin{pmatrix} -1 \\ -e^t \end{pmatrix} dt \]

\[ = \Phi(t) \int \frac{1}{2} e^{-8t} \begin{pmatrix} e^{3t} & e^{3t} \\ -e^{5t} & e^{5t} \end{pmatrix} \begin{pmatrix} -1 \\ -e^t \end{pmatrix} dt \]

\[ = \frac{1}{2} \Phi(t) \int \begin{pmatrix} -e^{-5t} - e^{-4t} \\ e^{-3t} - e^{-2t} \end{pmatrix} dt \]

\[ = \frac{1}{2} \Phi(t) \begin{pmatrix} \frac{1}{5} e^{-5t} + \frac{4}{3} e^{-4t} \\ -\frac{3}{5} e^{-3t} + \frac{2}{7} e^{-2t} \end{pmatrix} \]

\[ = \begin{pmatrix} -\frac{1}{8} e^t + \frac{4}{15} e^{-2t} \\ \frac{1}{8} e^t - \frac{1}{15} \end{pmatrix} \]

So the general solution is

\[ \mathbf{x}(t) = c_1 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} -e^{3t} \\ e^{3t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{8} e^t + \frac{4}{15} e^{-2t} \\ \frac{1}{8} e^t - \frac{1}{15} \end{pmatrix} \]
13. Find the fundamental matrix and use this to determine the general solution to
\[ \vec{x}'(t) = \begin{pmatrix} 5 & 4 \\ 1 & 8 \end{pmatrix} \vec{x} + \begin{pmatrix} t \\ -t \end{pmatrix} \]

**Solution: Problem 13:** The characteristic polynomial is
\[ \lambda^2 - 13\lambda + 36 = 0 \]
so the eigenvalues are \( \lambda_1 = 4 \) and \( \lambda_2 = 9 \). The corresponding eigenvectors are
\[ \xi_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
Hence the fundamental matrix is
\[ \Phi(t) = \begin{pmatrix} 4e^{4t} & e^{9t} \\ -e^{4t} & e^{9t} \end{pmatrix}. \]
A particular solution is given by
\[ \vec{x}_p(t) = \Phi(t) \int \Phi^{-1}(t) \begin{pmatrix} t \\ -t \end{pmatrix} dt \]
\[ = \Phi(t) \int \frac{1}{5} e^{-3t} \begin{pmatrix} e^{9t} & -e^{9t} \\ e^{4t} & 4e^{4t} \end{pmatrix} \begin{pmatrix} t \\ -t \end{pmatrix} dt \]
\[ = \frac{1}{5} \Phi(t) \int \begin{pmatrix} 2te^{4t} \\ -3te^{-9t} \end{pmatrix} dt \]
\[ = \frac{1}{5} \Phi(t) \begin{pmatrix} -\frac{2}{3}te^{-4t} - \frac{1}{3}e^{-4t} \\ \frac{1}{3}te^{-9t} + \frac{1}{27}e^{-9t} \end{pmatrix} \]
\[ = \begin{pmatrix} -\frac{1}{3}t + \frac{5}{3} \\ \frac{1}{6}t + \frac{5}{27} \end{pmatrix} \]
So the general solution is
\[ \vec{x}(t) = c_1 \begin{pmatrix} 4e^{4t} \\ -e^{4t} \end{pmatrix} + c_2 \begin{pmatrix} e^{9t} \\ e^{9t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}t + \frac{5}{3} \\ \frac{1}{6}t + \frac{5}{27} \end{pmatrix} \]