Part 1 - Solutions

1. Find the solution \( y(t) \) to

\[
y' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} y, \quad \text{with } y(0) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.
\]

Also, describe the behavior of the solution for large \( t \).

Solution: Problem 1:

The characteristic polynomial is

\[
\lambda^2 + 5\lambda + 4 = 0
\]

with roots \(-1\) and \(-4\). The eigenvector equation for \(-1\) is \(-x + y = 0\), and that for \(-4\) is \(x + 2y = 0\). The solutions are therefore

\[
y = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

For the initial conditions given, we solve

\[
\begin{align*}
c_1 + 2c_2 &= -3 \\
c_1 - c_2 &= 3,
\end{align*}
\]

with solutions \(c_1 = 1\) and \(c_2 = -2\). All solutions decay exponentially to 0 as both eigenvalues are negative.

2. Find the solution \( y(t) \) to

\[
y' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} y, \quad \text{with } y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

Solution: Problem 2:

The characteristic polynomial is

\[
\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0
\]

so \(\lambda = 2\) is an eigenvalue with multiplicity 2. The eigenvector equation is \(-2x + y = 0\), leading to an eigenvector \(\xi = (1, 2)\). We solve

\[(A - 2I)\eta = \xi\]

to get an equation \(-2x + y = 1\) for \(\eta\), or \(\eta = (0, 1)\). Therefore, the solutions are

\[
y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \left[ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].
\]

For the initial conditions we solve

\[
\begin{align*}
c_1 &= 1 \\
2c_1 + c_2 &= 3,
\end{align*}
\]
leading to $c_1 = 1$, $c_2 = 1$. All solutions grow exponentially fast as $t \to \infty$.

3. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$ 

Solution: Problem 3:
The characteristic polynomial is

$$\lambda^2 + 6\lambda + 13$$

leading to roots $\lambda_{1,2} = -3 \pm 2i$. The corresponding eigenvector equation is $-ix + y = 0$. Therefore the real solutions are the real and imaginary parts of

$$e^{-3+2it} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + i e^{-3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$ 

The solution is therefore,

$$y = c_1 e^{-3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$ 

The initial conditions give $c_1 = 1$, $c_2 = 2$. Solutions oscillate and decay.

4. Solve the following system of differential equations with initial conditions and sketch the solution. On your sketch, label the initial point, initial velocity vector and asymptotic direction.

$$\begin{pmatrix} x' \\ 4 \\ -7 \end{pmatrix} x = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solution: Problem 4:
The characteristic polynomial is

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$$

which leads to the roots $\lambda = -3$ with multiplicity 2. The eigenvector equation is therefore $4x - 4y = 0$ leading to the corresponding eigenvector is $\xi = (1, 1)$. We solve

$$(A + 3I)\eta = \xi$$

to get an equation $4x - 4y = 1$ for $\eta$, or $\eta = (1/4, 0)$. Therefore, the solutions are

$$y = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \left[ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right].$$

For the initial conditions we solve

$$c_1 + \frac{1}{4}c_2 = -1$$
$$c_1 = 1,$$
leading to $c_1 = 1, c_2 = -8$. All solutions decay exponentially fast as $t \to \infty$.

5. Analytically, find the solution $y(t)$ to the initial value problem

$$y' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Solution: Problem 5:

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

which leads to the root $\lambda = 1$ with multiplicity 2. The eigenvector equation is therefore $2y = 0$ leading to the corresponding eigenvector is $\xi = (1, 0)$. We solve

$$(A - I)\eta = \xi$$

to get an equation $2\eta = 1$ for $\eta$, or $\eta = (1, 1/2)$. Therefore, the solutions are

$$y = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \left[ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \right].$$

For the initial conditions we solve

$$c_1 + c_2 = 0$$

$$\frac{1}{2}c_2 = 1,$$

leading to $c_1 = -2, c_2 = 2$. All solutions grow exponentially fast as $t \to \infty$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Problem 4. A sketch labelling the initial point, initial velocity and asymptotic direction.}
\end{figure}
6. Show that all solutions of the system

\[ \mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} \]

approach zero as \( t \to \infty \) if and only if \( a + d < 0 \) and \( ad - bc > 0 \).

Solution: Problem 6:

The only way for all solutions of the system to approach zero as \( t \to \infty \) is if the eigenvalues of the system have negative real parts. Thus, we proceed to find conditions for the eigenvalues.

The characteristic polynomial is \( \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \) and therefore the eigenvalues are given by

\[ \lambda_{1,2} = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} = \frac{a + d}{2} \left( 1 \pm \sqrt{1 - \frac{4(ad - bc)}{(a + d)^2}} \right) \]

Observe that if \( (a + d) > 0 \) then the Real Part of \( \lambda_{1,2} \) is certainly positive because the square root quantity is either positive or imaginary and in either case, the real part of \( \lambda_{1,2} \) is positive. If \( (a + d) < 0 \) then the Real Part of \( \lambda_{1,2} < 0 \) provided \( 1 - \sqrt{1 - \frac{4(ad - bc)}{(a + d)^2}} > 0 \). This is not true if \( ad - bc < 0 \). Therefore, we require that \( a + d < 0 \) and \( ad - bc > 0 \) for both \( \lambda_{1,2} \) to be negative.

7. Match each direction field to a differential equation, or state that it does not match any of the equations. In each case, the point \((0, 0)\) is in the centre of the picture.
\[
\begin{align*}
(1) \quad x' &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x \\
(2) \quad x' &= \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} x \\
(3) \quad x' &= \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} x \\
(4) \quad x' &= \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} x \\
(5) \quad x' &= \begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} x
\end{align*}
\]

Solution: Problem 7:

The eigenvalues of the matrices determine how the pictures look. We have:

1. \( \lambda_1 = 1, \lambda_2 = 3 \). Two real positive values. Origin is an unstable node. \( \text{D} \).

2. \( \lambda_{1,2} = 2 \pm i \). Complex eigenvalue, positive real part. Origin is an unstable spiral. \( \text{A} \).

3. \( \lambda_{1,2} = \pm \sqrt{2}i \). Pure imaginary eigenvalues. Origin is a center point. \( \text{C} \).

4. \( \lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2} \). Positive and negative eigenvalues. Origin is a saddle. \( \text{E} \).

5. \( \lambda_{1,2} = -3 \pm \sqrt{2}i \). Complex eigenvalues, negative real part. Origin is a stable spiral. \( \text{B} \).