Show all relevant work for credit. You will be marked for your work and your answer as appropriate. Talking to other students about the problems is encouraged but you must submit your own work and identify who you worked with at the top of your assignment.

ALL WORK MUST BE STAPLED. WRITE YOUR NAME AND STUDENT NUMBER AT THE TOP OF THE FIRST PAGE. MESSY WORK WILL NOT BE GRADED.

1. Consider the initial value problem:

   \[ y'(t) = (t - 1)y(t) \quad y(0) = 1. \]

   (a) Solve analytically for \( y(t) \).

   (b) Use Euler’s method with \( h = 0.25 \) to estimate \( y(t) \) on \( 0 \leq t \leq 3 \). Repeat using \( h = 0.1 \) and \( h = 0.05 \). Submit one properly labelled plot showing the three Euler’s method solutions (three values of \( h \)).

   (c) Repeat the previous part using the Improved Euler method, again with \( h = 0.25, 0.1, 0.05 \). Submit one properly labelled plot showing the three Improved Euler’s method solutions (three values of \( h \)).

   (d) Finally, solve with \texttt{ode45} in Matlab/Octave (see previous homework for instructions). Submit a properly labelled plot of the solution.

   (e) Produce a table showing the absolute errors of the seven different numerical solutions compared to the analytical solution at \( t = 3 \).

   (f) For small \( h \), the absolute error at \( t = 3 \) with Euler’s method is commonly proportional to \( h \). Is that what you find? Similarly, the error of Improved Euler would be proportional to \( h^2 \) when \( h \) is small enough. Is that what you find? Is ODE45 more accurate than the other methods?

   **Note:** you can use Matlab for Euler and Improved Euler. Scripts for doing this are included with this homework. The scripts are set up with an example problem and a single choice of \( h \). You will need to edit the scripts to suit your purposes for these problems.

2. Repeat steps (b)-(d) of the previous problem for the following initial value problem:

   \[ y'(t) = \frac{2y(t) - 2}{\sin^3(t) + 2} \quad y(0) = 0. \]

   Solve on \( 0 \leq t \leq 3 \).
3. Logistic Model and Nondimensionalization

Nondimensionalization is a useful mathematical modelling tool to simplify equations and pick out the important parameters governing a system. We will consider the differential equation for logistic growth here.

Consider a population of fish in a lake, where \( P(T) \) will denote the population at a time \( T \) (measured in months) from now. We'll assume that right now there are \( P_0 \) fish in the lake. We will model the rate of change of \( P \) as proportional to \( P(K - P) \), or as an equation,

\[
\frac{dP}{dT} = rP(K - P) \quad \text{with} \quad P(0) = P_0.
\]

(a) Explain the meaning of \( K \) in words.

(b) Since the units of \( K \) must be number of fish (otherwise we couldn’t subtract \( P \) from \( K \)), we can define a new variable \( p = P/K \) which measures the fish population in units of \( K \). Show that this results in the ODE

\[
\frac{dp}{dT} = \mu p(1 - p)
\]

subject to \( p(0) = P_0/K \equiv p_0 \). What is \( \mu \)? What are the units of \( \mu \)?

(c) For what \( \sigma \) does the change of variables \( t = T/\sigma \) result in the further simplification to

\[
\frac{dp}{dt} = p(1 - p)
\]

subject to \( p(0) = p_0 \)?

(d) Solve analytically for \( p(t) \).

(e) Use Matlab to plot the slope field for the ODE. Superimpose example solutions with \( p_0 > 1, p_0 = 1, 0 < p_0 < 1 \). Label your plot properly.

4. Solve the following exact equations. Implicit general solutions are OK if you cannot solve for \( y(x) \). Hint: you may need to use a simple integrating factor \( u(x) \) or \( u(y) \) for some of these (see Lebl pages 59-61).

(a) \( (2xy + x^2) + (x^2 + y^2 + 1)\frac{dy}{dx} = 0 \)

(b) \( e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0 \)

(c) \( 2 \sin(y) + x \cos(y) \frac{dy}{dx} = 0 \)

5. A ball of mass \( m \) falls from rest from a height \( h \) towards the ground. We assume that the ball is acted upon by a constant gravitational force and by an opposing frictional force, which is proportional to the square of the velocity. Thus, the velocity \( v = v(t) \) (with \( v > 0 \) if the ball is falling downwards) satisfies

\[
m\frac{dv}{dt} = mg - kv^2, \quad v(0) = 0,
\]

where \( k > 0 \) is a constant.
(a) What is the terminal velocity \( \lim_{t \to \infty} v(t) \)?

(b) Calculate the velocity at any time \( t \) before the ball hits the ground. Hint: use partial fractions.

6. Consider the second order homogeneous equation

\[
ay''(t) + by'(t) + cy(t) = 0
\]

where \( b, c \) are real constants. This equation is equivalent to a 2x2 system of differential equations for
\[
\vec{x} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}.
\]
Find that system.

7. Solve the following system of differential equations with initial conditions and sketch the solution by hand (of course you can use Matlab to verify that the key features of your sketch are correct). On your sketch, label the initial point, initial velocity vector and asymptotic direction as \( t \to \infty \).

\[
x' = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]