**HOMEWORK 2 SOLUTIONS: MATH 215/255**  
Due in class on Friday, October 6th

1. Consider the initial value problem:

   \[ y'(t) = (t - 1)y(t) \quad y(0) = 1. \]

   (a) Solve analytically for \( y(t) \).

   (b) Use Euler’s method with \( h = 0.25 \) to estimate \( y(t) \) on \( 0 \leq t \leq 3 \). Repeat using \( h = 0.1 \) and \( h = 0.05 \). Submit one properly labelled plot showing the three Euler’s method solutions (three values of \( h \)).

   (c) Repeat the previous part using the Improved Euler method, again with \( h = 0.25, 0.1, 0.05 \). Submit one properly labelled plot showing the three Improved Euler’s method solutions (three values of \( h \)).

   (d) Finally, solve with \texttt{ode45} in Matlab/Octave (see previous homework for instructions). Submit a properly labelled plot of the solution.

   (e) Produce a table showing the absolute errors of the seven different numerical solutions compared to the analytical solution at \( t = 3 \).

   (f) For small \( h \), the absolute error at \( t = 3 \) with Euler’s method is commonly proportional to \( h \). Is that what you find? Similarly, the error of Improved Euler would be proportional to \( h^2 \) when \( h \) is small enough. Is that what you find? Is ODE45 more accurate than the other methods?

   **Note:** you can use Matlab for Euler and Improved Euler. Scripts for doing this are included with this homework. The scripts are set up with an example problem and a single choice of \( h \). You will need to edit the scripts to suit your purposes for these problems.

   **Solution: Problem 1:**

   (a) \( y'(t) = (t - 1)y(t) \) is a separable differential equation. Separating and integrating yields \( y(t) = Ce^{\frac{t^2}{2} - t}. \)

      Using the initial condition to solve for \( C \) gives \( C = 1 \) and the solution is \( y(t) = e^{\frac{t^2}{2} - t}. \)

   (b) In order to get Euler’s method script to work you only need to define the stepsize \( h \) and the appropriate number of steps \( N \), as well as changing \( f = @(t,y) \) to be the RHS of our differential equation, and \( y_{exact} \) to be your analytical solution. I have included a sample script that runs the three step sizes.

```matlab
clear all; close all;
% set start and end times for the problem
endtime = 3;
starttime = 0;
initial_condition = 1; % this should be y(starttime) for the problem

% RHS function
f=@(t,y) (t-1)*y;

% for this example problem, we can also solve the problem exactly - % you should try it - and this allows us to compare the solutions
```
y_exact=@(t) exp(t.^2/2-t);
hold on;
ezplot(y_exact,[starttime endtime])
for h=[0.25 0.1 0.05]
    N=(endtime-starttime)/h; % choose number of Euler steps
    % now set the first elements of y euler and t equal to their starting values
    y_euler(1)=initial_condition;
    t(1)=starttime;
    % this loop implements the Euler method described in class
    for k=1:N
        t(k+1)=t(k)+h;
        y_euler(k+1)=y_euler(k)+h*f(t(k),y_euler(k));
    end
    % now plot the solution
    plot(t,y_euler)
    end
    % finally output the final-time error between the improved
    % Euler solution and the exact solution. Note that Matlab vector indices
    % start at 1, so y[i] is valid from i=1 to i=N+1 and i=N+1 corresponds to
    % endtime
    abs(y_euler(N+1)-y_exact(endtime))
end

(c) Change $y_{euler}$ to $y_{IE}$ and the line with the difference equation to read
\[ y_{IE}(k+1) = y_{IE}(k) + \left( \frac{h}{2} \right) \left( f(t(k), y_{IE}(k)) + f(t(k+1), y_{IE}(k) + h \cdot f(t(k), y_{IE}(k))) \right); \]

(d) To use ode45 we can run the following script:

```matlab
% Problem 1d
% Use ode45 to solve y'=(t-1)*y

% Define RHS y'=f(t,y)
func = @(t,y) (t-1)*y;

% Use ode45 to solve ode45(function, time, IC)
[t,y] = ode45(func, [0,3],1);

% Plot solution
plot(t,y,'linewidth',1)
title('Numerical Solution of dy/dt=t*y')
xlabel('t')
ylabel('y')

% finally output the final-time error between the improved Euler solution and the exact solution. Note that Matlab vector indices start at 1, so y[i] is valid from i=1 to i=N+1 and i=N+1 corresponds to endtime
y_exact=@(t) exp(t.ˆ2/2-t);
abs(y(end)-y_exact(3))
```

(e) Here are the error values:

<table>
<thead>
<tr>
<th>Method</th>
<th>h</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.25</td>
<td>2.1808</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>1.1048</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.6065</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.1469</td>
</tr>
<tr>
<td>Imp. Euler</td>
<td>0.1</td>
<td>0.0261</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.0068</td>
</tr>
<tr>
<td>ode45</td>
<td>-</td>
<td>2.2978e-05</td>
</tr>
</tbody>
</table>

(f) The Euler errors are roughly proportional to \( h \) and the improved Euler is proportional to \( h^2 \). The Matlab routine ode45 has the smallest error.

2. Repeat steps (b)-(d) of the previous problem for the following initial value problem:

\[ y'(t) = \frac{2y(t) - 2}{\sin^4(t) + 2} \quad y(0) = 0. \]

Solve on \( 0 \leq t \leq 3. \)

Solution: Problem 2:
(b) Omit the exact solution from the script. In the code only change the function and the initial condition.

(c) Same for the improved Euler script.

(d) In the ode45 script change the function and the initial condition.
3. Logistic Model and Nondimensionalization

Nondimensionalization is a useful mathematical modelling tool to simplify equations and pick out the important parameters governing a system. We will consider the differential equation for logistic growth here.

Consider a population of fish in a lake, where \( P(T) \) will denote the population at a time \( T \) (measured in months) from now. We’ll assume that right now there are \( P_0 \) fish in the lake. We will model the rate of change of \( P \) as proportional to \( P(K - P) \), or as an equation,

\[
\frac{dP}{dT} = rP(K - P) \quad \text{with} \quad P(0) = P_0.
\]

(a) Explain the meaning of \( K \) in words.

(b) Since the units of \( K \) must be number of fish (otherwise we couldn’t subtract \( P \) from \( K \)), we can define a new variable \( p = P/K \) which measures the fish population in units of \( K \). Show that this results in the ODE

\[
\frac{dp}{dT} = \mu p(1 - p)
\]

subject to \( p(0) = P_0/K \equiv p_0 \). What is \( \mu \)? What are the units of \( \mu \)?

(c) For what \( \sigma \) does the change of variables \( t = T/\sigma \) result in the further simplification to

\[
\frac{dp}{dt} = p(1 - p)
\]

subject to \( p(0) = p_0 \)?

(d) Solve analytically for \( p(t) \).

(e) Use Matlab to plot the slope field for the ODE. Superimpose example solutions with \( p_0 > 1, p_0 = 1, 0 < p_0 < 1 \). Label your plot properly.

Solution: Problem 3:

(a) \( K \) represents the carrying capacity of the lake. When \( K > P \), we see that \( \frac{dP}{dT} > 0 \) and therefore the population grows. When \( K < P \), \( \frac{dP}{dT} < 0 \) and the population declines. When \( K = P \), \( \frac{dP}{dT} = 0 \). Therefore, the population cannot exceed this value and if \( P_0 > K \) then the population will decrease to \( K \).

(b) If \( p = P/K \) then \( P = Kp \) and the equation becomes

\[
\frac{d(Kp)}{dT} = K \frac{dp}{dT} = rKp(K - Kp) = rK^2p(1 - p).
\]

Dividing the equation by \( K \) gives us

\[
\frac{dp}{dT} = rKp(1 - p)
\]

so \( \mu = rK \). The initial condition states \( Kp(T = 0) = P_0 \) so \( p(T = 0) = P_0/K = p_0 \). As \( p = P/K \) has no units (i.e. \( [p] = [P]/[K] = \text{fish/fish} = 1 \)), the dimensions on the left are that of \( 1/T \), which has units of months\(^{-1}\). On the right, \( p(1 - p) \) is also dimensionless, so the units of \( \mu \) must be months\(^{-1}\).
(c) Letting \( t = T/\sigma \) means that, by the chain rule, \( \frac{d}{dT} p = \frac{d}{dt} p \times \frac{dt}{dT} = \frac{1}{\sigma} \frac{dp}{dt} \). Thus the ODE under this new change of variables is
\[
\frac{1}{\sigma} \frac{dp}{dt} = \mu p(1 - p)
\]
so if \( \frac{1}{\sigma} = \mu \) (i.e. \( \sigma = 1/\mu \)) we have
\[
\frac{dp}{dt} = p(1 - p).
\]
Note, we could have guessed something of this form by looking at the units of \( \mu \) being 1/time, so a time-scale \( \sigma \) should be proportional to \( 1/\mu \).

(d) We have \( \frac{dp}{dt} = p(1 - p) \) so that upon separating variables and beginning to integrate, we have \( \int \frac{dp}{p(1-p)} = \int dt \). The left-hand side can be solved with partial fractions:
\[
\int \frac{1}{p(1-p)} = \frac{A}{p} + \frac{B}{1-p} \implies 1 = A(1-p) + Bp.
\]
Setting \( p = 1 \) yields \( B = 1 \) and setting \( p = 0 \) gives \( A = 1 \) so
\[
\int \frac{dp}{p(1-p)} = \int \left( \frac{1}{p} + \frac{1}{1-p} \right) dp = \ln|p| - \ln|1-p| + C = \ln(p/(1-p)) + C.
\]
Therefore \( \ln|p/(1-p)| = t + C \) and \( p/(1-p) = De^t \). At \( t = 0 \), \( p = p_0 \) so \( \frac{p_0}{1-p_0} = D \). We can also rearrange \( p/(1-p) = De^t \) to find \( p = \frac{De^t}{D+1} = \frac{p_0}{1+e^{t}\frac{p_0}{1-p_0}} \). Technically \( D \) is not defined when \( p_0 = 1 \) but in that case \( \frac{dp}{dt} = 0 \) for all time so \( p(t) = 1 \) is the solution (see the slope field). The solution above without the \( D \) works for all \( p_0 \) initial values.

(e) % Problem 6 - The logistic model and nondimensionalization
% p' = p(1-p)
% Define RHS, p' = f(p,t)
func = @(t,p) p*(1-p);
% Define Initial Condition p0 > 1
p0 = 1.5;
% Run ode45
[t,y] = ode45(func, [0,10],p0);
% Plot solution
hold on
plot(t,y,'linewidth',3)
% Define Initial Condition p0 = 1
p0 = 1;
% Run ode45
[t,y] = ode45(func, [0,10],p0);
% Plot solution
4. Solve the following exact equations. Implicit general solutions are OK if you cannot solve for \( y(x) \). Hint: you may need to use a simple integrating factor \( u(x) \) or \( u(y) \) for some of these (see Lebl pages 59-61).

(a) \( (2xy + x^2) + (x^2 + y^2 + 1) \frac{dy}{dx} = 0 \)

(b) \( e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0 \)

(c) \( 2 \sin(y) + x \cos(y) \frac{dy}{dx} = 0 \)
Solution: Problem 4:

(a) \((2xy + x^2) + (x^2 + y^2 + 1) \frac{dy}{dx} = 0\)

We first test to see if this is an exact differential equation. We see that \(M(x, y) = 2xy + x^2\) and \(N(x, y) = x^2 + y^2 + 1\). It is easy to see that,

\[ M_y = 2x = N_x \]

and therefore the equation is exact. Thus, there exists a \(\psi(x, y)\) such that

\[ \psi_x(x, y) = 2xy + x^2 \]

\[ \psi_y(x, y) = x^2 + y^2 + 1 \]

Integrating the first equation, we obtain

\[ \psi(x, y) = x^2y + \frac{x^3}{3} + h(y) \]

Setting \(\psi_y(x, y) = N\) gives

\[ \psi_y(x, y) = x^2 + h'(y) = x^2 + y^2 + 1 \]

Thus, \(h'(y) = y^2 + 1\) and \(h(y) = \frac{y^3}{3} + y\). Substituting this in gives,

\[ \psi(x, y) = x^2y + \frac{x^3}{3} + \frac{y^3}{3} + y \]

and the solution is given implicitly by

\[ x^2y + \frac{x^3}{3} + \frac{y^3}{3} + y = c \]

(b) \(e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0\)

The equation is exact as, \(M_y = 3y^2 = N_x\). The implicit solution is given by

\[ xy^3 + e^x = c \]

(c) \(2 \sin(y) + x \cos(y) \frac{dy}{dx} = 0\)

Solution 1: This can made into an exact equation by multiplying through by \(x\). We now have \(M_y = 2x \cos(y) = N_x\). See Solution 2 for the final answer.

Solution 2: We can separate variables easily obtaining

\[ \frac{\cos(y)}{\sin(y)} dy = \frac{-2}{x} dx \]

Integrating both sides and minor simplification yields the solution

\[ y = \sin^{-1}(\frac{c}{x^2}) \]

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5. A ball of mass \( m \) falls from rest from a height \( h \) towards the ground. We assume that the ball is acted upon by a constant gravitational force and by an opposing frictional force, which is proportional to the square of the velocity. Thus, the velocity \( \dot{v} = v(t) \) (with \( v > 0 \) if the ball is falling downwards) satisfies

\[
\frac{m}{k} \frac{dv}{dt} = mg - kv^2, \quad v(0) = 0,
\]

where \( k > 0 \) is a constant.

(a) What is the terminal velocity (\( \lim_{t \to \infty} v(t) \))?

(b) Calculate the velocity at any time \( t \) before the ball hits the ground. Hint: use partial fractions.

Solution: Problem 5:

(a) In the event of terminal velocity, acceleration is zero. Thus, \( \frac{dv}{dt} = 0 \). Using this in the ODE, yields \( mg = kv^2 \). We then solve this for \( v \) and get \( v = \sqrt{\frac{mg}{k}} \).

(b) Separate variables and use partial fractions to get

\[
\frac{dv}{\frac{mg}{k} - v^2} = \frac{k}{m} dt \quad \left( \frac{1}{\sqrt{\frac{mg}{k} + v}} + \frac{1}{\sqrt{\frac{mg}{k} - v}} \right) = 2 \sqrt{\frac{mg}{k}} \frac{kt}{m}.
\]

Integrating the equation and using \( v(0) = 0 \), we get

\[
\ln \left( \frac{\sqrt{\frac{mg}{k} + v}}{\sqrt{\frac{mg}{k} - v}} \right) = 2 \sqrt{\frac{kg}{m}} t.
\]

Solving for \( v \), we obtain

\[
v = \sqrt{\frac{mg}{k}} \left( \frac{1 - e^{-\sigma t}}{1 + e^{-\sigma t}} \right), \quad \sigma = 2 \sqrt{\frac{kg}{m}}. \tag{\ast}
\]

Note that when \( t \to \infty \), we obtain the answer from part (a).

6. Consider the second order homogeneous equation

\[
ay''(t) + by'(t) + cy(t) = 0
\]

where \( b, c \) are real constants. This equation is equivalent to a 2x2 system of differential equations for \( \vec{x} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \). Find that system.

Solution: Problem 6:

Let

\[
\vec{x} = \begin{pmatrix} y \\ y' \end{pmatrix}
\]
then
\[
x' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} -\frac{b}{a} y' - \frac{c}{a} y \\ 0 - \frac{b}{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} y' \\ y'' \end{pmatrix}
\]
so
\[
x' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} x.
\]

7. Solve the following system of differential equations with initial conditions and sketch the solution by hand (of course you can use Matlab to verify that the key features of your sketch are correct). On your sketch, label the initial point, initial velocity vector and asymptotic direction as \( t \to \infty \).

\[
x' = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Solution: Problem 7:

First calculate the eigenvalues and corresponding eigenvectors of the matrix.

\[
0 = \det \begin{pmatrix} 2 - \lambda & -1 \\ 4 & -3 - \lambda \end{pmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2),
\]

so the eigenvalues are \( \lambda_1 = 1, \lambda_2 = -2 \). The eigenvector for \( \lambda_1 \) solves

\[
\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

so is \( v_1 = (1, 1) \). The second eigenvector solves

\[
\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

so is \( v_2 = (1, 4) \). Therefore, we can write the solution as

\[
x(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.
\]

For the initial conditions given we solve

\[
c_1 + c_2 = 1
\]
\[
c_1 + 4c_2 = 0,
\]

with solutions \( c_1 = 4/3, c_2 = -1/3 \). Therefore the solution is

\[
x(t) = \frac{4}{3} e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{3} e^{-2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.
\]
Figure 3: Figure for Problem 7. The initial point is the bottom left corner. The direction of the initial velocity vector is in blue with a diamond. The solution is the yellowish line. The asymptotic direction is along the purple line, which corresponds to the unstable eigenvector $(1, 1)$. 