Show all relevant work for credit. You will be marked for your work and your answer as appropriate. Talking to other students about the problems is encouraged but you must submit your own work and identify who you worked with at the top of your assignment.

All work must be stapled. Write your name and student number at the top of the first page. Messy work will not be graded.

1. Consider the initial value problem:

\[ y'(t) = ty(t) \quad y(0) = 1. \]

(a) Solve analytically for \( y(t) \).

(b) Use Euler’s method with \( h = 0.25 \) to estimate \( y(t) \) on \( 0 \leq t \leq 1 \). Repeat using \( h = 0.05 \) and \( h = 0.01 \). Submit one properly labelled plot showing the three Euler’s method solutions (three values of \( h \)).

(c) Repeat the previous part using the Improved Euler method, again with \( h = 0.25, 0.05, 0.01 \). Submit one properly labelled plot showing the three Improved Euler’s method solutions (three values of \( h \)).

(d) Finally, solve with \texttt{ode45} in Matlab/Octave (see previous homework for instructions). Submit a properly labelled plot of the solution.

(e) Produce a table showing the absolute errors of the seven different numerical solutions compared to the analytical solution at \( t = 1 \).

(f) For small \( h \), the absolute error at \( t = 1 \) with Euler’s method should be proportional to \( h \). Is that what you find? Similarly, the error of Improved Euler should be proportional to \( h^2 \) when \( h \) is small enough. Is that what you find? Is ODE45 more accurate than the other methods?

Note: you can use Matlab for Euler and Improved Euler. Scripts for doing this are included with this homework. The scripts are set up with an example problem and a single choice of \( h \). You will need to edit the scripts to suit your purposes for these problems.

Solution: Problem 1:

(a) \( y'(t) = ty(t) \) is a separable differential equation. Separating and integrating yields \( y = ce^{\frac{t^2}{2}} \). Using the initial condition to solve for \( c \) gives \( c = 1 \) and the solution is \( y(t) = e^{\frac{t^2}{2}} \).

(b) In order to get Euler’s method script to work you only need to change, \( N \) for the appropriate number of Euler steps, \( f = @(t,y) \) to be the RHS of our differential equation, and \( y_\text{exact} \) exact to be your analytical solution. For \( h = 0.25, N = 4; h = 0.05, N = 20; h = 0.01, N = 100 \). I have included a sample script for \( h = 0.25 \).
% this script implements Euler's method for the problem y' = f(t,y)
% below. The time values are stored in the vector t with corresponding y
% values stored in vector y_euler.

%clear all; close all;

N = 4; % choose number of Euler steps

% set start and end times for the problem
endtime = 1;
starttime = 0;
initial_condition = 1; % this should be y(starttime) for the problem

% RHS function (set to an example y' = y + t)
f = @(t,y) t*y;

% define the step size h
h = (endtime - starttime) / N;

% now set the first elements of y_euler and t equal to their starting values
y_euler(1) = initial_condition;
t(1) = starttime;

% this loop implements the Euler method described in class
for k = 1:N
    t(k+1) = t(k) + h;
    y_euler(k+1) = y_euler(k) + h * f(t(k), y_euler(k));
end

% now plot the solution
plot(t, y_euler)

% for this example problem, we can also solve the problem exactly -
% you should try it - and this allows us to compare the solutions
y_exact = @(t) exp(t^2/2);
hold on;
ezplot(y_exact, [0 1])

% finally output the final-time error between the improved
% Euler solution and the exact solution. Note that Matlab vector indices
% start at 1, so y[i] is valid from i = 1 to i = N+1 and i = N+1 corresponds to
% endtime
abs(y_euler(N+1) - y_exact(endtime))
Figure 1: Problem 1b. Euler’s Method for different $h$ values plotted with the analytical solution.

(c) The same changes are made to the Improved Euler script as in part (b) (script omitted because of similarity).

(d) % Problem 1d
% Use ode45 to solve $y'=t\cdot y$

% Define RHS $y'=f(t,y)$
func = @(t,y) t*y;

% Use ode45 to solve ode45(function, time, IC)
[t,y] = ode45(func, [0,1],1);

% Plot solution
plot(t,y,'lineweight',3)
title('Numerical Solution of $\frac{dy}{dt}=t\cdot y$')
xlabel('t')
ylabel('y')

% finally output the final-time error between the improved
% Euler solution and the exact solution. Note that Matlab vector indices
% start at 1, so $y[1]$ is valid from $i=1$ to $i=N+1$ and $i=N+1$ corresponds to
% endtime
y_exact=@(t) exp(t^2/2);
abs(y(end)-y_exact(1))
(f) For Euler, it is clear that the error is on the order of $h$. For Improved Euler, however we see the error is on the order of $h^3$. ode45 is the most accurate method used.

2. Repeat all steps of the previous problem for the following initial value problem:

$$y'(t) = \sin(t) + y(t) \quad y(0) = 0.$$  

Solution: Problem 2:

(a) $y'(t) = \sin(t) + y(t)$ can be solved using the integrating factor $\mu(t) = e^{-\int dt} = e^{-t}$. This yields

$$y = ce^t - \frac{\sin(t)}{2} - \frac{\cos(t)}{2}.$$  

Using the initial condition to solve for $c$ gives $c = 1/2$ and the solution is

$$y = \frac{1}{2}(e^t - \sin(t) - \cos(t)).$$

Figure 2: Problem 1c. Improved Euler’s Method for different $h$ values plotted with the analytical solution.
Numerical Solution of \( \frac{dy}{dt} = ty \)

Figure 3: Problem 1d. Numerical solution using ode45.

(b) In order to get Euler’s method script to work you only need to change, \( N \) for the appropriate number of Euler steps, \( f = @(t,y) \) to be the RHS of our differential equation, the initial condition, and \( y_{\text{exact}} \) to be your analytical solution. For \( h = 0.25, N = 4; h = 0.05, N = 20; h = 0.01, N = 100 \). Only the relevant changes to the script are shown in the box.

\[
N=4; \ % \text{choose number of Euler steps}
\text{initial\_condition} = 0; \ % \text{this should be y(starttime) for the problem}
f=@(t,y) \sin(t) + y;
y_{\text{exact}}=@(t) 0.5*(\exp(t)-\sin(t)-\cos(t));
\]

(c) The same changes are made to the Improved Euler script as in part (b) (script omitted because of similarity).

(d)

\[
\text{% Problem 2d}
\text{% Use ode45 to solve y'=\sin(t) + y}
\text{% Define RHS y'=f(t,y)}
\text{func = @(t,y) \sin(t) + y}
\text{% Use ode45 to solve ode45(function, time, IC)}
[t,y] = ode45(func, [0,1],0);
\text{% Plot solution}
\]
Figure 4: Problem 2b. Euler’s Method for different h values plotted with the analytical solution.

Figure 5: Problem 2c. Improved Euler’s Method for different h values plotted with the analytical solution.
Figure 6: Problem 2d. Numerical solution using ode45.

```matlab
plot(t,y,'linewidth',3)
title('Numerical Solution of dy/dt=t*y')
xlabel('t')
ylabel('y')

% finally output the final-time error between the improved
% Euler solution and the exact solution. Note that Matlab vector indices
% start at 1, so y[i] is valid from i=1 to i=N+1 and i=N+1 corresponds to
% endtime
y_exact=@(t) 0.5*(exp(t)-sin(t)-cos(t));
abs(y(end)-y_exact(1))
```

<table>
<thead>
<tr>
<th>Method</th>
<th>$h$</th>
<th>Error</th>
</tr>
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</tr>
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<td>0.0011</td>
</tr>
<tr>
<td>ode45</td>
<td>0.01</td>
<td>4.6066e-05</td>
</tr>
</tbody>
</table>

(f) For Euler, it is clear that the error is on the order of $h$. For Improved Euler, we see the error is on the order of $h^2$. ode45 is the most accurate method used.
3. Consider the second order homogeneous equation

\[ y''(t) + by'(t) + cy(t) = 0 \]

where \( b, c \) are real constants. This equation is equivalent to a 2x2 system of differential equations for \( \vec{x} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \). Find that system.

**Solution: Problem 3:**

Let

\[ \vec{x} = \begin{pmatrix} y' \\ y \end{pmatrix} \]

then

\[ \vec{x}' = \begin{pmatrix} y'' \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ -by - cy \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} \]

so

\[ \vec{x}' = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \vec{x} \]

4. Solve the following system of differential equations with initial conditions and sketch the solution by hand (of course you can use Matlab to verify that the key features of your sketch are correct). On your sketch, label the initial point, initial velocity vector and asymptotic direction as \( t \to \infty \).

\[ \vec{x}' = \begin{pmatrix} 2 & -2 \\ 4 & -7 \end{pmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

**Solution: Problem 4:**
The eigenvalues and corresponding eigenvectors of the matrix are \( \lambda_1 = -6, \lambda_2 = 1 \) and \( v_1 = (1, 4), v_2 = (2, 1) \). Therefore, we can write the solution as

\[ \vec{x}(t) = c_1 e^{-6t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

For the initial conditions given we solve

\[ c_1 + 2c_2 = 0 \]
\[ 4c_1 + c_2 = 1, \]

with solutions \( c_1 = 2/7, c_2 = -1/7 \). Therefore the solution is

\[ \vec{x}(t) = \frac{2}{7} e^{-6t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{1}{7} e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

5. Consider the initial value problem

\[ \frac{dy}{dt} - 6y = -9e^{2t} \]

with \( y(0) = A \).
Figure 7: Figure for Problem 4. The initial point is the small circle. The direction of the initial velocity vector is indicated as a black arrow. The solution is the thick red line. The asymptotic direction is along the green line, which corresponds to the unstable eigenvector $(2, 1)^T$. The background vector field (blue) corresponding to the system of differential equations is also plotted.
(a) Find the analytic solution.

(b) For what values of $A$ does the above solution tend to $\infty$, 0, or $-\infty$ as $t \to \infty$?

(c) Use Matlab to produce a single plot showing three example solutions exhibiting each possible behaviour of the equation as $t \to \infty$. Label the plot to indicate the value of $A$ used for each solution.

Solution: Problem 5:

(a) Using the integrating factor $\mu(t) = e^{-6t} = e^{-6t}$ we get the answer $y(t) = c_1 e^{6t} + \frac{9}{4} e^{2t}$. Applying the initial condition $y(0) = A$ gives $c_1 = A - \frac{9}{4}$. Plugging this in gives the solution

$$y(t) = (A - \frac{9}{4}) e^{6t} + \frac{9}{4} e^{2t}$$

(b) Because both exponentials have positive exponents, both of these terms go to $\infty$ as $t \to \infty$. Therefore, for $y \to \infty$ both terms must be positive. Therefore, $A \geq 9/4$. For $y \to -\infty$ the $e^{6t}$ dominates and therefore we need a negative sign on this term. This implies $A < 9/4$. There is no such $A$, such that $y \to 0$ as $t \to \infty$.

(c) ```matlab
% Problem 5

% Define RHS, y' = f(y,t)
func = @(t,y) 6*y - 9*exp(2*t);

% Define Initial Condition such that y->infty (y0 >= 9/4)
y0 = 3;

% Run ode45
[t,y] = ode45(func, [0,2],y0);

% Plot solution
hold on
plot(t,y,'linewidth',3)

% Define Initial Condition such that y -> - infty (y < 9/4)
y0 = 1;

% Run ode45
[t,y] = ode45(func, [0,2],y0);

% Plot solution
plot(t,y,'g','linewidth',3)

% Add labels to plot
xlabel('t')
ylabel('y')
legend('y_0 = 3','y_0 = 1')
```
6. Logistic Model and Nondimensionalization

Nondimensionalization is a useful mathematical modelling tool to simplify equations and pick out the important parameters governing a system. We will consider the differential equation for logistic growth here. Consider a population of fish in a lake, where \( P(T) \) will denote the population at a time \( T \) (measured in months) from now. We’ll assume that right now there are \( P_0 \) fish in the lake. We will model the rate of change of \( P \) as proportional to \( P(K - P) \), or as an equation,

\[
\frac{dP}{dT} = rP(K - P) \quad \text{with} \quad P(0) = P_0.
\]

(a) Explain the meaning of \( K \) in words.

(b) Since the units of \( K \) must be number of fish (otherwise we couldn’t subtract \( P \) from \( K \)), we can define a new variable \( p = P/K \) which measures the fish population in units of \( K \). Show that this results in the ODE

\[
\frac{dp}{dT} = \mu p(1 - p)
\]

subject to \( p(0) = P_0/K \equiv p_0 \). What is \( \mu \)? What are the units of \( \mu \)?

(c) For what \( \sigma \) does the change of variables \( t = T/\sigma \) result in the further simplification to

\[
\frac{dp}{dt} = p(1 - p)
\]

subject to \( p(0) = p_0 \).

(d) Solve analytically for \( p(t) \).
(c) Use Matlab to plot the slope field for the ODE. Superimpose example solutions with \( p_0 > 1, p_0 = 1, 0 < p_0 < 1 \). Label your plot properly.

Solution: Problem 6:

(a) \( K \) is the carrying capacity of the population. When \( K > P \), we see that \( \frac{dP}{dT} > 0 \) and therefore the population grows. When \( K < P \), \( \frac{dP}{dT} < 0 \) and the population declines. When \( K = P \), \( \frac{dP}{dT} = 0 \). Therefore, the population cannot exceed this value and if \( P_0 > K \) then the population will decrease to \( K \).

(b) If \( \frac{p}{K} \) then \( P = Kp \) and the equation becomes

\[
\frac{d(Kp)}{dT} = K \frac{dp}{dT} = rKp(K - Kp) = rK^2 p(1 - p).
\]

Dividing the equation by \( K \) gives us

\[
\frac{dp}{dT} = rKp(1 - p)
\]

so \( \mu = rK \). The initial condition states \( Kp(T = 0) = P_0 \) so \( p(T = 0) = P_0/K = p_0 \). As \( p = P/K \) has no units (i.e. \( [p] = [P]/[K] = \text{fish/fish} = 1 \)), the dimensions on the left are that of \( 1/T \), which has units of months\(^{-1}\). On the right, \( p(1 - p) \) is also dimensionless, so the units of \( \mu \) must be months\(^{-1}\).

(c) Letting \( t = T/\sigma \) means that, by the chain rule, \( \frac{d}{dT}p = \frac{d}{dt}p \times \frac{dt}{dT} = \frac{1}{\sigma} \frac{dp}{dt} \). Thus the ODE under this new change of variables is

\[
\frac{1}{\sigma} \frac{dp}{dt} = \mu p(1 - p)
\]

so if \( 1/\sigma = \mu \) (i.e. \( \sigma = 1/\mu \)) we have

\[
\frac{dp}{dt} = p(1 - p).
\]

Note, we could have guessed something of this form by looking at the units of \( \mu \) being 1/time, so a time-scale \( \sigma \) should be proportional to \( 1/\mu \).

(d) We have \( \frac{dp}{dt} = p(1 - p) \) so that upon separating variables and beginning to integrate, we have \( \int \frac{dp}{p(1 - p)} = \int dt \). The left-hand side can be solved with partial fractions:

\[
\int \frac{dp}{p(1 - p)} = \int \left( \frac{1}{p} + \frac{1}{1 - p} \right) dp = \ln |p| - \ln |1 - p| + C = \ln(p/(1 - p)) + C.
\]

Setting \( p = 1 \) yields \( B = 1 \) and setting \( p = 0 \) gives \( A = 1 \) so

\[
\int \frac{dp}{p(1 - p)} = \left( \frac{1}{p} + \frac{1}{1 - p} \right) dp = \ln |p| - \ln |1 - p| + C = \ln(p/(1 - p)) + C.
\]

Therefore \( \ln|p/(1 - p)| = t + C \) and \( p/(1 - p) = De^t \). At \( t = 0 \), \( p = p_0 \) so \( \frac{p_0}{1 - p_0} = D \). We can also rearrange \( p/(1 - p) = De^t \) to find \( p = \frac{De^t}{De^t + 1} = \frac{p_0}{p_0 + (1 - p_0)e^{-t}} \). Technically \( D \) is not defined when \( p_0 = 1 \) but in that case \( \frac{dp}{dt} = 0 \) for all time so \( p(t) = 1 \) is the solution (see the slope field). The solution above without the \( D \) works for all \( p_0 \) initial values.
% Problem 6 - The logistic model and nondimensionalization
% p' = p(1-p)

% Define RHS, p' = f(p,t)
func = @(t,p) p*(1-p);

% Define Initial Condition p0 > 1
p0 = 1.5;

% Run ode45
[t,y] = ode45(func, [0,10], p0);

% Plot solution
hold on
plot(t,y,'linewidth',3)

% Define Initial Condition p0 = 1
p0 = 1;

% Run ode45
[t,y] = ode45(func, [0,10], p0);

% Plot solution
plot(t,y,'g','linewidth',3)

% Define Initial Condition p0 < 1
p0 = 0.5;

% Run ode45
[t,y] = ode45(func, [0,10], p0);

% Plot solution
plot(t,y,'r','linewidth',3)

% Plot the slope field
dirfield(func,0:1:10,0:0.2:2)

% Add labels to plot
xlabel('t')
ylabel('p')
legend('p_0 = 1.5','p_0 = 1','p_0 = 0.5')
7. Solve the following exact equations. Implicit general solutions are OK if you cannot solve for $y(x)$. Hint: you may need to use an integrating factor for some of these (see Lebl pages 59-61).

(a) $(2xy + x^2) + (x^2 + y^2 + 1)\frac{dy}{dx} = 0$

(b) $x^5 + y^5 \frac{dy}{dx} = 0$

(c) $e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0$

(d) $e^{xy} + ye^{xy} \frac{dy}{dx} = 0$

(e) $2\sin(y) + x\cos(y) \frac{dy}{dx} = 0$

Solution: Problem 7:

(a) $(2xy + x^2) + (x^2 + y^2 + 1)\frac{dy}{dx} = 0$

We first test to see if this is an exact differential equation. We see that $M(x,y) = 2xy + x^2$ and $N(x,y) = x^2 + y^2 + 1$. It is easy to see that,

$$M_y = 2x = N_x$$

and therefore the equation is exact. Thus, there exists a $\psi(x,y)$ such that

$$\psi_x(x,y) = 2xy + x^2$$
$$\psi_y(x,y) = x^2 + y^2 + 1$$
Integrating the first equation, we obtain

\[ \psi(x, y) = x^2 y + \frac{x^3}{3} + h(y) \]

Setting \( \psi_y(x, y) = N \) gives

\[ \psi_y(x, y) = x^2 + h'(y) = x^2 + y^2 + 1. \]

Thus, \( h'(y) = y^2 + 1 \) and \( h(y) = \frac{y^3}{3} + y \). Substituting this in gives,

\[ \psi(x, y) = x^2 y + \frac{x^3}{3} + \frac{y^3}{3} + y \]

and the solution is given implicitly by

\[ x^2 y + \frac{x^3}{3} + \frac{y^3}{3} + y = c \]

(b) \( x^5 + y^5 \frac{dy}{dx} = 0 \)

Following the procedure in the previous problem, we easily see that \( M_y = 0 = N_x \). The implicit solution is given by

\[ y^6 + x^6 = c. \]

(c) \( e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0 \)

The equation is exact as, \( M_y = 3y^2 = N_x \). The implicit solution is given by

\[ x^3 y^3 + e^x = c. \]

(d) \( e^{xy} + \frac{y}{x} e^{xy} \frac{dy}{dx} = 0 \)

**Solution 1:** The goal is to turn this into an exact differential equation. Begin by multiplying through by \( e^{-xy} \) yielding

\[ 1 + \frac{y}{x} \frac{dy}{dx} = 0. \]

Now multiply through by \( x \) to obtain

\[ x + \frac{dy}{dx} = 0. \]

This is now an exact differential equation as \( M_y = 0 = N_x \). The implicit solution is given by

\[ x^2 + y^2 = c. \]

**Solution 2:** Begin by multiplying through by \( e^{-xy} \). We now have the differential equation \( 1 + \frac{y}{x} \frac{dy}{dx} = 0 \). Separating variables yields \( ydy = -xdx \). Integrating both sides and rearranging gives the implicit solution

\[ x^2 + y^2 = c. \]
(c) $2 \sin(y) + x \cos(y) \frac{dy}{dx} = 0$

**Solution 1:** This can made into an exact equation by multiplying through by $x$. We now have $M_y = 2x \cos(y) = N_x$. See Solution 2 for the final answer.

**Solution 2:** We can separate variables easily obtaining

$$\frac{\cos(y)}{\sin(y)} dy = -\frac{2}{x} dx.$$

Integrating both sides and minor simplification yields the solution

$$y = \sin^{-1}\left(\frac{c}{x^2}\right).$$

8. A ball of mass $m$ falls from rest from a height $h$ towards the ground. We assume that the ball is acted upon by a constant gravitational force and by an opposing frictional force, which is proportional to the square of the velocity. Thus, the velocity $v = v(t)$ (with $v > 0$ if the ball is falling downwards) satisfies

$$m \frac{dv}{dt} = mg - kv^2, \quad v(0) = 0,$$

where $k > 0$ is a constant.

(a) What is the terminal velocity ($\lim_{t \to \infty} v(t)$) ?

(b) Calculate the velocity at any later time $t$ before the ball hits the ground. Hint: use partial fractions.

**Solution: Problem 8:**

- (a) In the event of terminal velocity, acceleration is zero. Thus, $\frac{dv}{dt} = 0$. Using this in the ODE, yields $mg = kv^2$. We then solve this for $v$ and get $v = \sqrt{\frac{mg}{k}}$.

- (b) Separate variables and use partial fractions to get

$$\frac{dv}{\frac{mg}{k} - v^2} = \frac{k}{m} dt \left( \frac{1}{\sqrt{\frac{mg}{k} + v}} + \frac{1}{\sqrt{\frac{mg}{k} - v}} \right) = 2 \sqrt{\frac{mg \, kt}{k \, m}}.$$

Integrating the equation and using $v(0) = 0$, we get

$$\ln \left( \frac{\sqrt{\frac{mg}{k} + v}}{\sqrt{\frac{mg}{k} - v}} \right) = 2 \sqrt{\frac{kg}{m}} t.$$

Solving for $v$, we obtain

$$v = \sqrt{\frac{mg}{k}} \left( \frac{1 - e^{-\sigma t}}{1 + e^{-\sigma t}} \right), \quad \sigma \equiv 2 \sqrt{\frac{kg}{m}}. \quad (*)$$

Note that when $t \to \infty$, we obtain the answer from part (a).