4. Consider the linear system of ODEs with real parameter $\beta$:

$$\begin{align*}
\frac{dx}{dt} &= x + y \\
\frac{dy}{dt} &= x + \beta y
\end{align*}$$

Suppose the initial point is not at the origin, $(x(0), y(0)) \neq (0, 0)$.
For what values of $\beta$ and initial conditions $(x(0), y(0))$ is it possible for the solutions to approach $(0, 0)$ as $t \to \infty$? Explain your answer carefully.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & \beta \end{bmatrix}$$

$$\lambda^2 - (1+\beta) \lambda + (\beta - 1) = 0 \quad \text{(1)}$$

$$\lambda = \frac{(1+\beta) \pm \sqrt{(1+\beta)^2 - 4(\beta-1)}}{2} \quad \text{(1)}$$

$$(1+\beta)^2 - 4(\beta-1) = 1+\beta^2 + 2\beta - 4\beta + 4 = (\beta-1)^2 + 4 > 0 \quad \text{(1)}$$

So $\lambda_1$ and $\lambda_2$ are real.

$$\lambda_1 = \frac{1+\beta + \sqrt{(1+\beta)^2 - 4(\beta-1)}}{2}$$

$$\lambda_2 = \frac{1+\beta - \sqrt{(1+\beta)^2 - 4(\beta-1)}}{2}$$

$\beta > 1$ : $\lambda_1, \lambda_2 > 0 \rightarrow \text{not possible}$

$-1 < \beta < 1$ : $\lambda_1 > 0$, $\lambda_2 < 0 \rightarrow \text{saddle point}$

$\beta < -1$ : $\lambda_1 > 0$, $\lambda_2 < 0 \rightarrow \text{saddle point}$
So, the only option to go to the origin is to have a saddle point.

\[
\sqrt{(1+\beta)^2 - 4(\beta - 1)} > 1 + \beta
\]

\[
(1+\beta)^2 - 4(\beta - 1) > (1+\beta)^2
\]

\[
\beta - 1 < 0 \quad \text{or} \quad \beta < 1
\]

So \((x(0), y(0))\) should be located on the zero direction \(\vec{n}_2\) (corresponding to \(\lambda_2\)) to go to the direction \(\vec{n}_2\).