① March 3

Midterm # 2 : March 12 in class

Taylor's Formula with Remainder

(Course Notes § 2.)

Last class we found the n^{th} degree Taylor Polynomial for $f(x)$ around $x = a$ using

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}}$$

Last class for $f(x) = \ln x$ around $a=1$.
We found,

$$T_2(x) = x - 1 - \frac{1}{2}(x-1)^2 .$$

We can approximate

$$\ln(1/2) = f(1/2) \simeq T_2(1/2) = -0.625 .$$

The true value is $\ln(1/2) = -0.693\ldots$

②

How do we know our approximation is good?

Today we will quantify the error in our approximation.

We write,

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{T_n(x)} + R_n(x)$$

where the function $R_n(x)$ is the remainder. It is the difference between our true function, $f(x)$, and our approximation, $T_n(x)$.

$$f(x) - T_n(x) = R_n(x).$$

We can't find $R_n(x)$ exactly. We can hope to bound the remainder (error).

We want to say: $R_n(x)$ is no bigger than blah.

For example: if we know that

$$|R_n(x)| \le 0.1 .$$

then we know

$$f(x) = T_n(x) \pm 0.1 .$$

The theorem we need is the Lagrange Remainder Formula.

Suppose $f(x)$ has $n+1$ derivatives. Consider the Taylor Polynomial around $x=a$ with remainder $R_n(x)$.

Then,
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

The proof comes from the Mean Value Theorem (next week). Skip proof.

④

Let's try to do this with our last example.

$f(x) = \ln x$ around $a = 1$.

$f(x) - T_2(x) = R_2(x)$, $T_2(1/2) = -0.625$.

$$R_2(x) = \frac{f'''(c)}{3!}(x-a)^3$$

for some c between x and a.

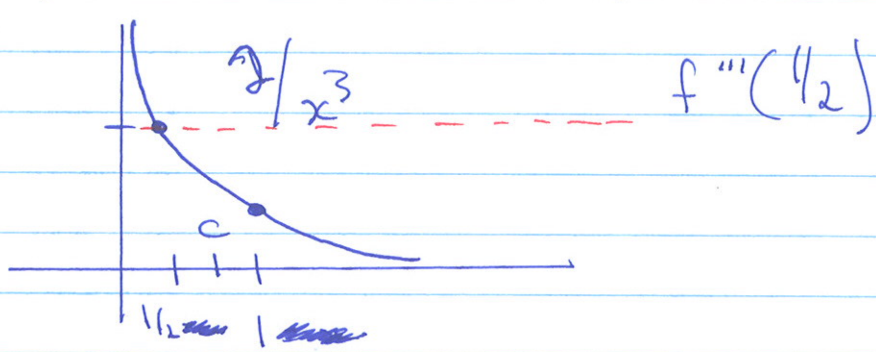
(here $x = 1/2$, $a = 1$.)

$f'(x) = 1/x$, $f''(x) = -1/x^2$.

$f'''(x) = +2/x^3$.

Clicker Q: Which value of c would be the worst?

→ A: $1/2$ C: 1
B: $3/4$ D: other.



(5)

$$\begin{array}{ccccc} \cancel{ex}\,x & & & & a \\ \shortparallel & & & & \shortparallel \\ \end{array}$$

So, if $\frac{1}{2} \le c \le 1$

$$|f'''(c)| \le |f'''(1/2)|$$
$$= 2/(1/2)^3 = 2/1/8 = 2 \cdot 8/1$$
$$= 16.$$

So, $|R_2(x)| \le \left|\frac{16(1/2 - 1)^3}{3!}\right|$

$$= \left|\frac{16(-1/2)^3}{3!}\right| = 2/3!$$
$$= 1/3.$$

So, $\ln(1/2) = -0.625 \pm 1/3$.

You can imagine if you found $R_3(x)$ or $R_4(x)$ the error will get smaller.

⑥

Let's formulate everything nicely.

We have $R_n(x) = \dfrac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}$

The goal is to find some number M so that

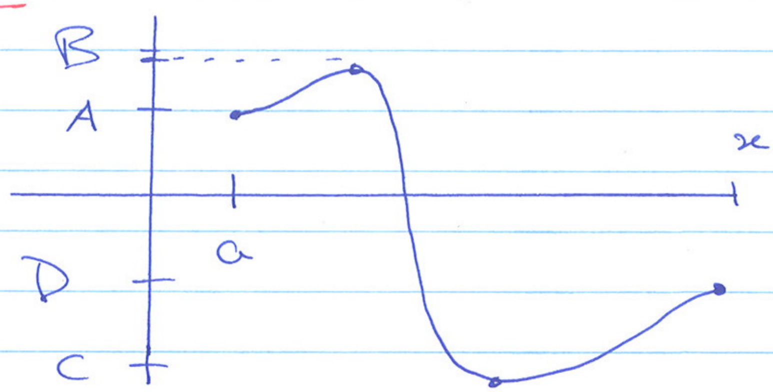
$$|f^{(n+1)}(c)| \le M \quad \text{for } c \text{ between } a \text{ and } x.$$

In this way,

$$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!}.$$

In the previous example $M = 16$.

Clicker Q! $f^{(5)}(x)$ is shown.



Which M should you take in

$$|R_4(x)| \le \frac{M}{5!}|x-a|^5 \ ?$$

C Take $M = C$.

Example: Recall $f(x) = \sin x = \underbrace{x - \frac{x^3}{3!}}_{T_3(x)} + R_3(x)$,

around $x = 0$.

So, $\sin(0.5) \simeq 0.5 - \frac{(0.5)^3}{3!}$

(0.5 radians is about $28.6°$)

$$\frac{1}{2} - \frac{1}{8 \cdot 3!} = 0.47916\ldots$$

Estimate $|R_3(x)| \le M\frac{|x-a|^4}{4!}$.

with $x = 0.5$ and $a = 0$.

Find M.

$$|f^{(4)}(c)| \le M$$

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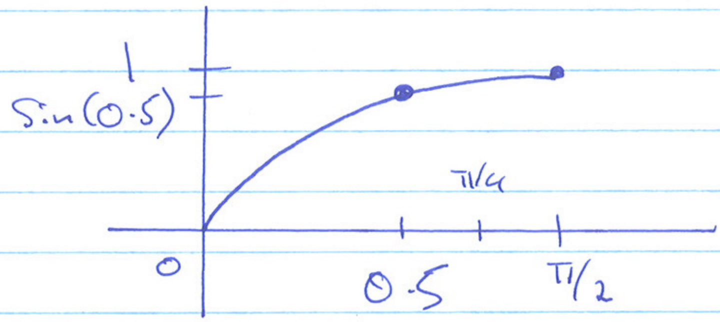
$f^{(4)}(x) = \sin x$.

We want to bound $\sin(c)$ on $[0, 0.5]$.



$|\sin x| = |f^{(4)}(x)| \leq \sin(0.5)$ on $[0, 0.5]$.

We could take $M = 1$ because

$|f^{(4)}(x)| \leq \sin(0.5) < 1 = M$

Note $\sin(\pi/4) = 1/\sqrt{2}$

$\sin(0.5) < 1/\sqrt{2}$.

So, $M = 1/\sqrt{2}$ is better.

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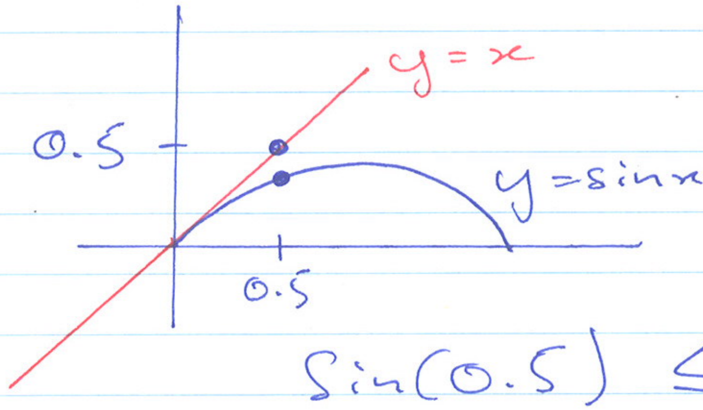
An even better M is



$\sin(0.5) \le 0.5$.

Take $M = 1/2$.

So $|R_3(x)| \le \frac{1}{2} \frac{|1/2 - 0|^4}{4!}$

$= 0.00130\ldots$

So $\sin(0.5) = 0.47916\ldots \pm 0.00131$

$0.477 < \sin(0.5) < 0.481$

(True value: $\sin(0.5) = 0.4794\ldots$)

Example: Find the third degree Taylor polynomial for $f(x) = \sqrt{x}$ around $x = 1$.
Use it to approximate $\sqrt{2}$.
Estimate the error in this approximation.

Try this on your own.

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

$$f^{(4)}(x) = \frac{-15}{16} x^{-7/2}$$

$$T_3(2) = 1.4375$$

$|f^{(4)}(c)| \leq M$ for c in $[1, 2]$

$M = 15/16$.

So. $$|R_3(x)| \leq M \frac{|2-1|^4}{4!} = \frac{15}{16} \frac{1}{4!}$$