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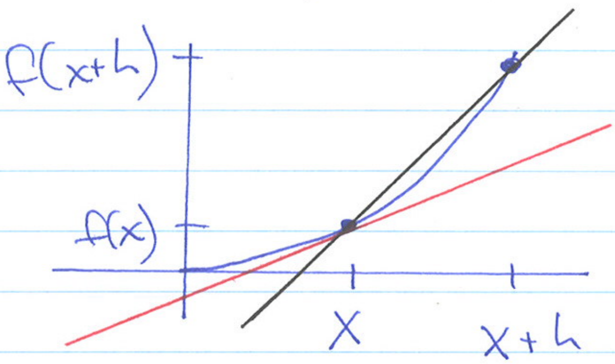
§ 2.8 The Derivative as a Function

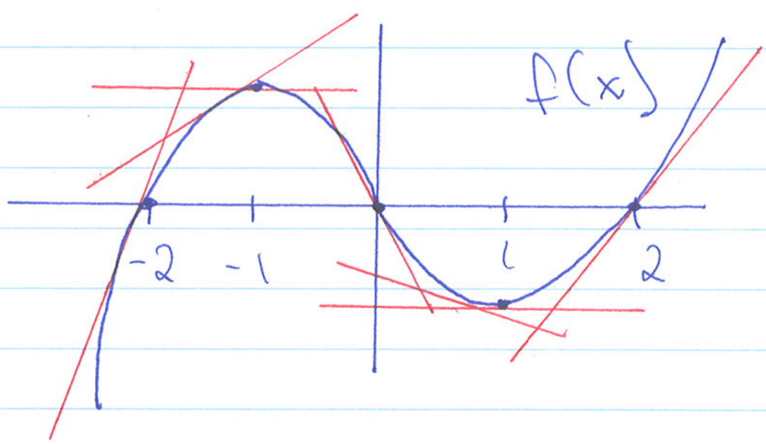
Last class we discovered how to find the derivative using limits.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Slope of the tangent line to f(x) at point x.

Clicker Q! Which of the following graphs is the derivative of f(x)



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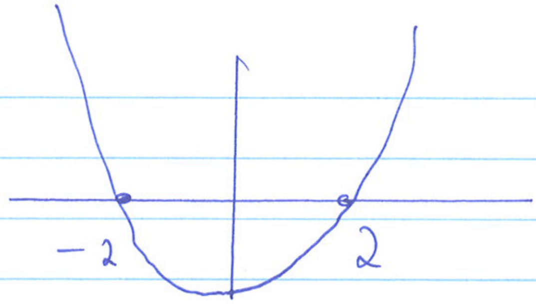
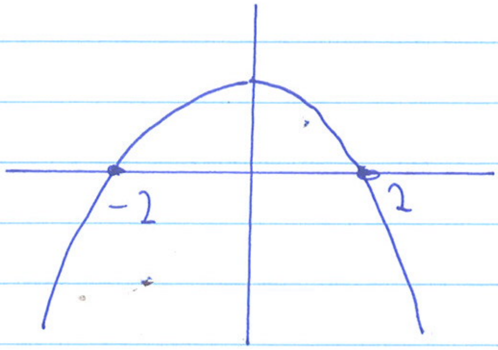
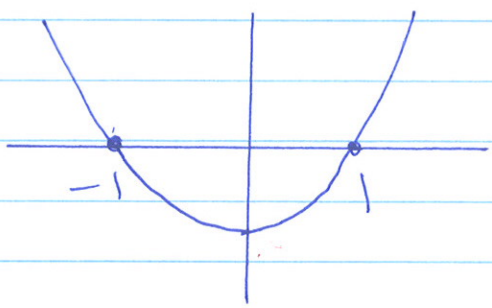
A :



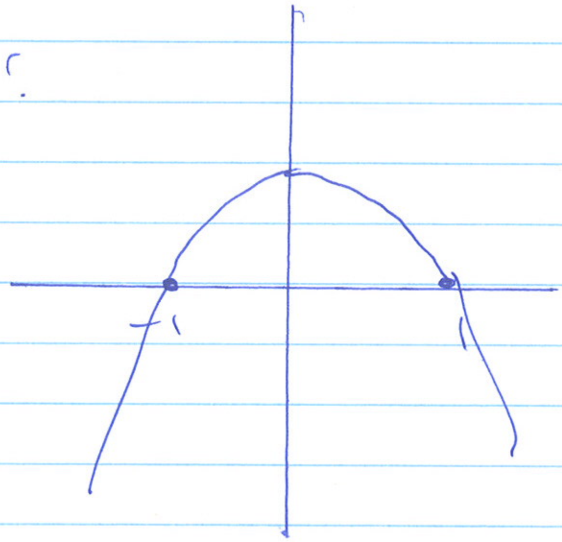
B :



C :



D :



Example: Find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}.$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Note: The domain of $f(x)$ is $\{x \geq 0\}$. but the domain of $f'(x)$ is $\{x > 0\}$.

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$$\left(\frac{(h)^{1/2}}{h^1} = h^{-1/2} \right)$$

What is happening at $x = 0$?

$$f'(0) = \lim_{h \to 0} \frac{\sqrt{0+h} - \sqrt{0}}{h}$$

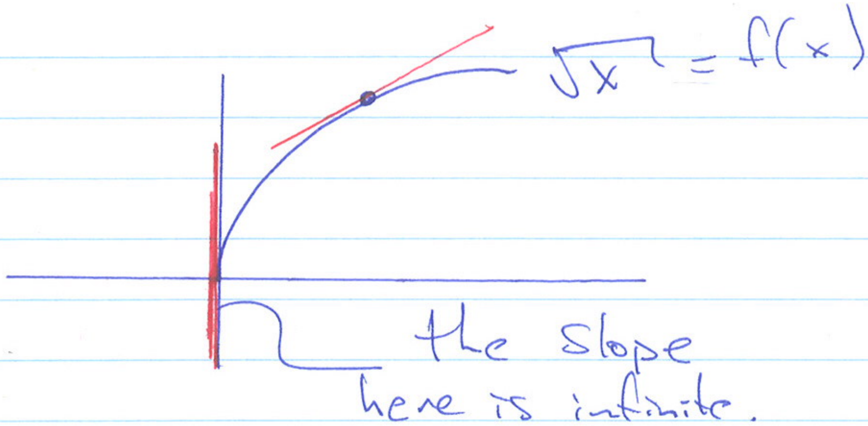
$$= \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty$$

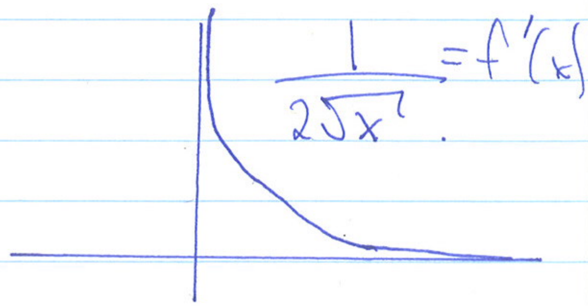
The limit does not exist ("equals infinity") and so the derivative of $f(x)$ at $x = 0$ does not exist.

Definition: A function is differentiable at $x = a$ if the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

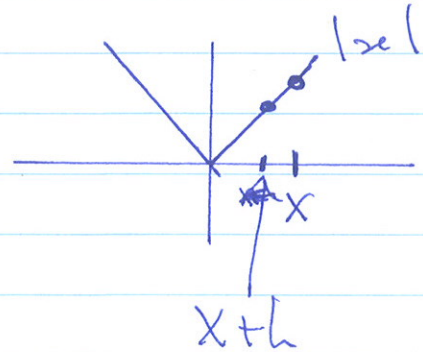
exists.





Example! Find the derivative of $f(x) = |x|$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



For $x > 0$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{|x+h| - |x|}{h}.$$

$$= \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1$$

$$= 1.$$

The slope for $x > 0$ is 1.

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For $x<0$: $\lim_{h\to 0} \frac{|x+h| - |x|}{h}$.

$= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h}$

$= \lim_{h\to 0} \frac{-h}{h} = -1$

The slope for $x<0$ is -1.

But, when $x=0$:

$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{|0+h| - |0|}{h}$

$= \lim_{h\to 0} \frac{|h|}{h} \leftarrow$ D.N.E.

One sided limits are not equal. So the full limit D.N.E.

$$\begin{cases} \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \\ \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1. \end{cases}$$

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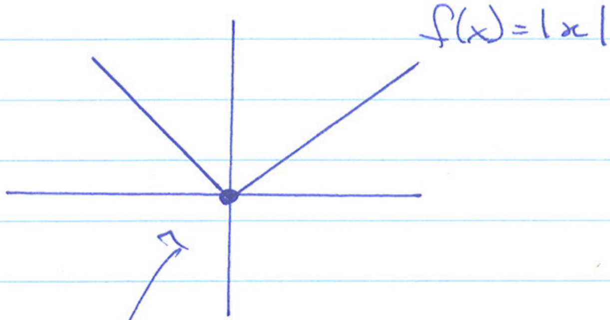
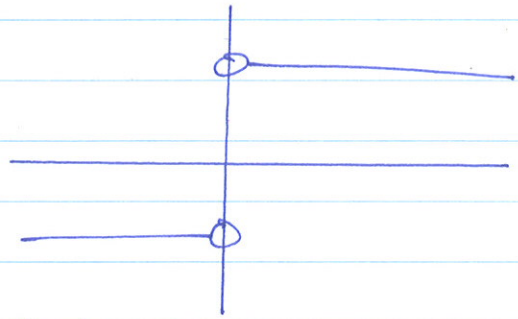
$f'(0)$ D.N.E. $f(x)$ is <u>not</u> differentiable at $x = 0$.





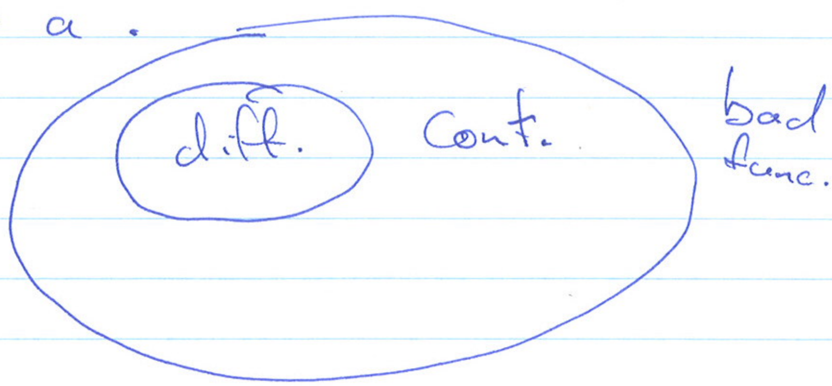
$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

We cannot draw a tangent line at the corner $(x=0)$.

Is $|x|$ continuous? → A: Yes
B: No.

Is $|x|$ differentiable? Not at $x = 0$.

Theorem: (p.158) If f is differentiable at $x = a$ then f is continuous at $x = a$.



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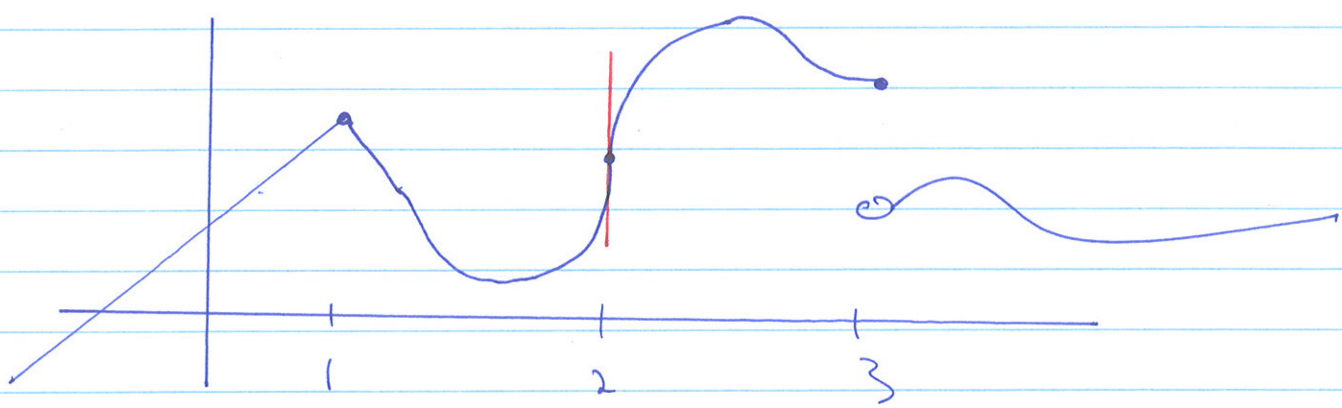
Clicker Q: At how many points is $f(x)$ not differentiable?



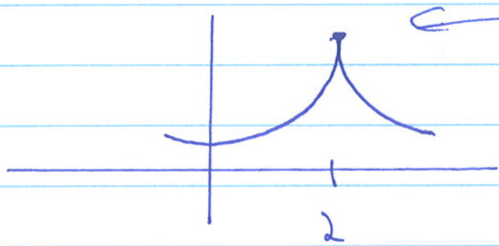
A: 0
B: 1
C: 2
→ D: 3
E: 4.

f has a corner at $x = 1$

f has "infinite slope" at $x = 2$,

f is discontinuous at $x = 3$.

Also:



← cusp
not differentiable.

Notation:

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f$$

↑ Newton-ish ↑ Leibniz

$$f'(x) = Df = D_x f \,.$$

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$$f'(a) = \left.\frac{df}{dx}\right|_{x=a}.$$

Higher Derivatives:

$$\big(f'(x)\big)' = f''(x).$$

$$\frac{d}{dx}\left(\frac{d}{dx} f\right) = \frac{d^2 f}{dx^2}.$$

$$\vdots$$

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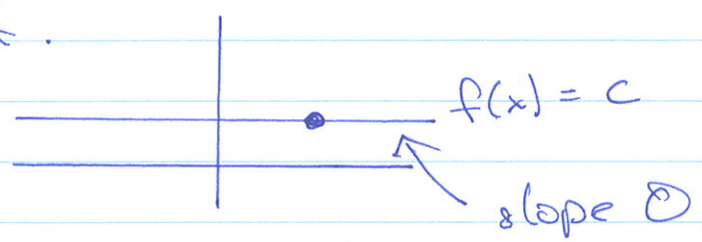
§ 3.1 Derivatives of Polynomials and Exponentials.

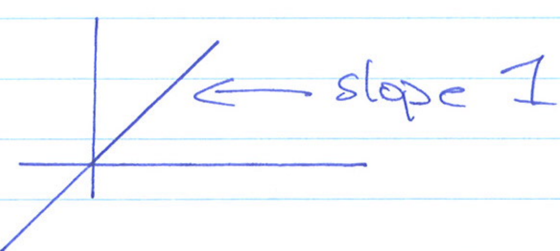
Polynomials:

constant.

$$f(x) = c$$
$$f'(x) = 0$$



$$f(x) = x$$
$$f'(x) = 1$$



$$\left.\begin{aligned} f(x) &= x^2. \\ f'(x) &= 2x^1 \end{aligned}\right\}$$ Saw last class with definition.

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In general $\boxed{\frac{d}{dx}(x^r) = r x^{r-1}}$

Power Rule (p. 173 – 175).

Examples: $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$f(x) = x^{-2/5}$, $f'(x) = -\frac{2}{5} x^{-7/5}$.

You can use the following rules to break down a function (like limit laws)

Theorem: (p. 176 – 177) Let c be a constant. Let f and g be differentiable. Then,

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}.$$

$$\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}.$$

$$\frac{d}{dx}(cf) = c \cdot \frac{df}{dx}$$

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Example: $f(x) = 4x^3 - x^{-2} + \pi^4$.

$f'(x) = 12x^2 + 2x^{-3} + 0$.

constant.