1

Midterm #2: March 12 (in class)

§ 3.9 Related Rates

Example: A spherical balloon is filling with air at a rate of $100 \text{cm}^3/\text{s}$. How fast is the radius increasing when the diameter is 50 cm?

V - volume

r - radius

We have the rate of change of the volume. $\frac{dV}{dt} = 100$

We want to relate this to the rate of change of the radius, $\frac{dr}{dt}$.

$$V = \frac{4}{3}\pi r^3 . \quad \text{(Volume of sphere)}$$

Let's use chain rule.

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$$

2

Find $\frac{dr}{dt}$ when $r = 25$.

100

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{4\pi r^2}\,\frac{dV}{dt}.$$

at $r = 25$: $$\frac{dr}{dt} = \frac{1}{4\pi(25\,\text{cm})^2} \cdot 100\ \text{cm}^3/\text{s}$$

$$= \frac{1}{25\pi}\ \text{cm/s}.$$

To solve this problem we went through some steps.

1. Read / Understand (picture)
2. Notation, Given / Required Rate.
3. Find Equation(s).
4. Chain Rule.
5. Solve / Substitute.
6. Reflect.

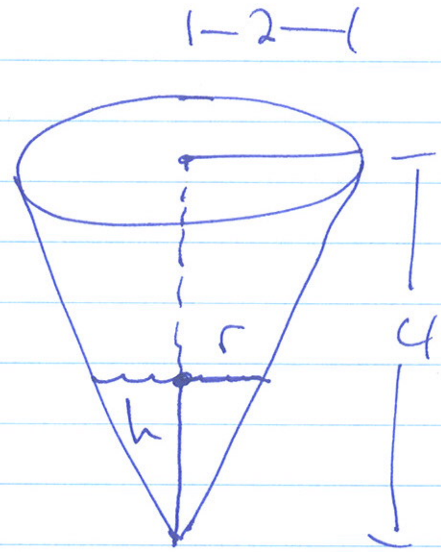
$V = \frac{1}{3}\pi r^2 h$

Example: A connical water tank has a base radius of 2m and a height of 4m. If water is being drained from the tank at a rate of $2\text{m}^3/\text{min}$, find the rate of change of the water level when the water is 3m deep.

A: Working B: Getting Somewhere C: Stuck.

(3)

1. Picture



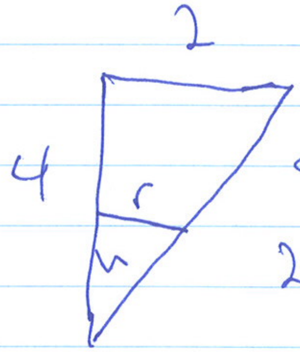
2. Notation h and r are functions of time.

Have: $\frac{dV}{dt}$

Need: $\frac{dh}{dt}$.

3. Equation $V = \frac{1}{3}\pi r^2 h$.

$r = \frac{h}{2}$



← Similar triangles.

$2 = \frac{4}{2} = \frac{h}{r} \Rightarrow r = \frac{h}{2}$.

④

We can eliminate r in favour of h.

So, $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{1}{3}\pi \frac{h^3}{4}$.

4. Chain Rule.

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}.$$

5. Solve/Sub.

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

$-2m^3/min$

At $h = 3m$ $\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot(-2)$

$= -\frac{8}{9\pi}$ m/min

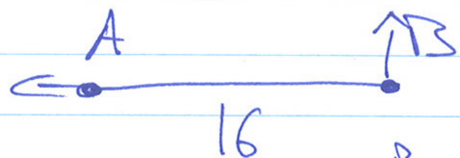
The ~~wat~~ height is decreasing at $\frac{8}{9\pi}$ m/min.

6. Reflect Negative.

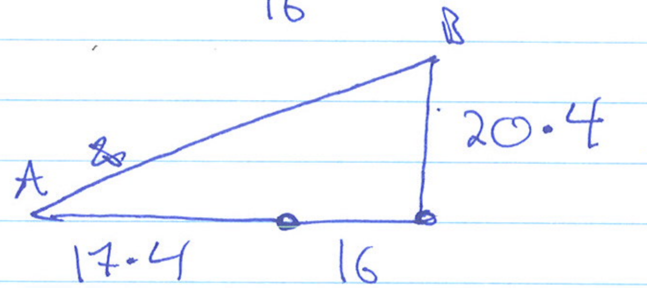
Example: At noon ship Alice is 16 nautical miles west of ship Bob.
Alice is sailing west at 17 knots.
Bob is sailing north at 20 knots.
How fast is the distance between them changing at 4pm?
1 knot is 1 nautical mile per hour.

1. Picture

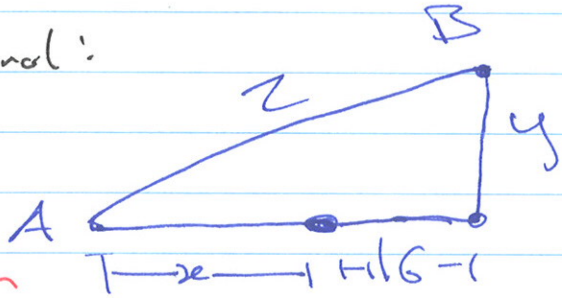
Noon:



4pm:



General:



2. Notation

y - distance B has traveled

x - distance A has traveled.

Want: $\frac{dz}{dt}$ Have: $\frac{dx}{dt}, \frac{dy}{dt}$.

3. Equation $(x+16)^2 + y^2 = z^2$.

4. Chain Rule.

$$\cancel{2} z \frac{dz}{dt} = \cancel{2}(x+16)\frac{dx}{dt} + \cancel{2} y \frac{dy}{dt}.$$

5. Solve. $\frac{dx}{dt} = 17$, $\frac{dy}{dt} = 20$.

at 4pm $\begin{cases} x = 17 \cdot 4 = 68 \\ y = 20 \cdot 4 = 80 \\ z = 116 \end{cases}$

$$z = \sqrt{(68+16)^2 + 80^2}.$$
$$= 116.$$

$$116 \cdot \frac{dz}{dt} = 84 \cdot 17 + 80 \cdot 20.$$

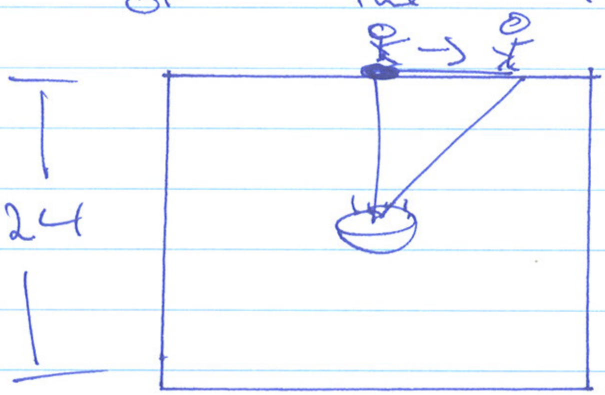
$$\frac{dz}{dt} = \ldots = \frac{757}{29} \text{ knots}$$

$$\simeq 26 \text{ knots}.$$

6. Reflect. Ans. reasonable, Bigger than $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

⑦

Example: A man walks along the edge of a square field with side length 24m. A spotlight is positioned in the middle of the field.



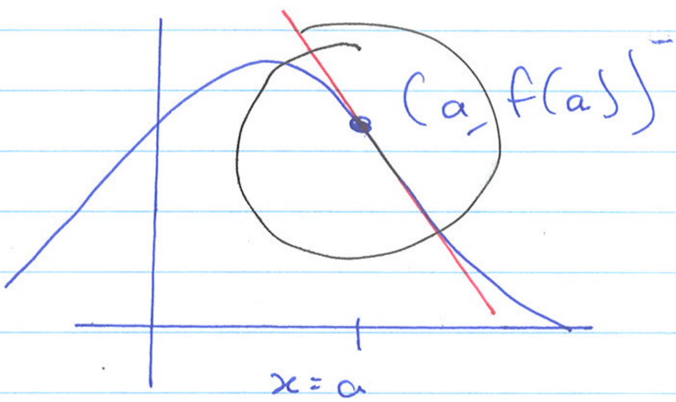
When he reaches the midpoint of the edge the light turns on. He runs at 2m/s, as the spotlight follows him. How fast is the spotlight rotating after 2.5 seconds.

Try this on your own.

8

§ 3.10 Linear Approximation.

We have spent a lot of time finding the equation of the tangent line at a point $x = a$.



The equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

or

$$y = f'(a)(x-a) + f(a).$$

Notice that close to $x = a$ the tangent line does a reasonable job approximating the function.

Let's call this line $L(x)$.

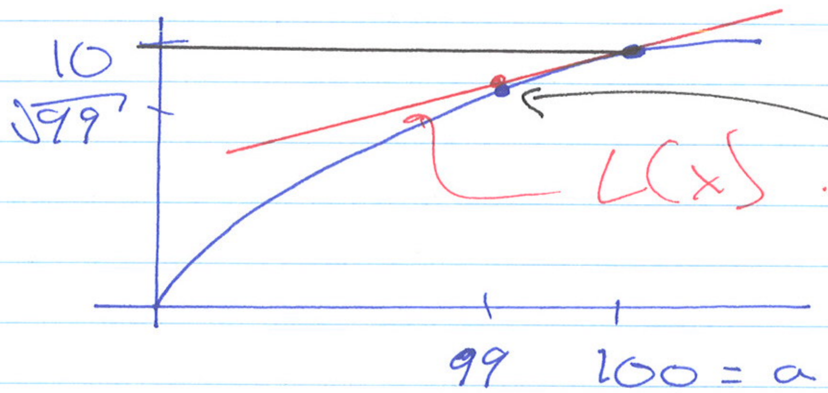
$$L(x) = f(a) + f'(a)(x-a).$$

$L(x)$ is the linear approximation or linearization of $f(x)$

Example: Approximate $\sqrt{99}$ using a linear approximation.

We know that $\sqrt{100} = 10$.
Our first guess would be $\sqrt{99} \simeq 10$.
We can do better with a linear approximation.

$f(x) = \sqrt{x}$



$L(99)$ and $f(99)$ are close.
$L(99)$ is a little bigger than $f(99) = \sqrt{99}$.

10.

Find $L(x) = f(a) + f'(a)(x-a)$

with $f(x) = \sqrt{x}$ $\quad a = 100.$

$f(a) = \sqrt{100} = 10.$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(a) = f'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}.$

$L(x) = 10 + \dfrac{1}{20}(x-100).$

Now $L(x)$ approximates $f(x)$.

So, $\sqrt{99} = f(99) \simeq L(99).$

$L(99) = 10 + \dfrac{1}{20}(99-100)$

$= 10 - \dfrac{1}{20} = 9.95.$

Actual answer: $9.94987\ldots$

Over Estimate.