I. Curves

- Parametrization: \( \vec{r}(t) = \langle \ldots, \ldots, \ldots \rangle \) \( a \leq t \leq b \)

Diagram:

- Orientation = Direction, usually from \( \vec{r}(a) \) to \( \vec{r}(b) \)
- Geometry (curvature, ...)
- Motion (acceleration, ...)
- Integral over curves:

\[
\int_C \int f \, ds = \int_a^b f(\mathbf{r}(t)) \left| \mathbf{r}'(t) \right| \, dt
\]

in particular if \( f = 1 \): get arclength

\[
s(t) = \int_a^t \left| \mathbf{r}'(u) \right| \, du \quad \text{distance along curve from } a \text{ to } t.
\]

reparameterize by arclength:
reparameterize by arc length:

\[ s = \text{something with } t \]

\[ \begin{align*}
  t &= \text{something with } s \\
  \therefore \hat{r}(s) &= \hat{r}(t)
\end{align*} \]

\(-\) vector fields: work integrals

\[ \oint_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T} \, ds \]

\[ = \int_C P \, dx + Q \, dy \ (\ + \ R \, dz) \]
\[
\int_{t=a}^{b} F(\tilde{r}(t)) \cdot \tilde{r}'(t) \, dt
\]

\[
\int_{t=a}^{b} P(\tilde{r}(t)) x'(t) + Q(\tilde{r}(t)) y'(t) \, dt
\]

How to compute these?

- Integrals of functions:
  Use parameterization to compute directly.
Integrals of vector fields:

a) Is \( \vec{F} \) conservative?

If yes: find \( f \) with \( \nabla f = \vec{F} \)

and use the fundamental
then of line integrals.
or maybe use path independence.

b) Is \( C \) a closed curve (=loop)?

(check: \( \vec{r}(a) = \vec{r}(b) \))

Use Green's Theorem or Stoke's Thm.
Compute directly

Advanced tricks:

1) If $C$ is not a closed curve, you might be able to add a curve (often the line from $\tilde{r}(b)$ to $\tilde{r}(a)$) to make it closed.
\( F \) not conservative, but close, can try to write \( F = F_1 + F_2 \)

\[ \uparrow \text{very simple.} \]

**How to check if \( F \) is conservative?**

\[ \exists f : \nabla f = F \]

\[ Q_x - P_y = 0 \]

in 3D: \( F \) conservative \( \Rightarrow \) \( \text{curl } F = 0 \)
How to compute a potential?

want $\nabla \Phi = \vec{F} = \langle P, Q \rangle$

\[ f_x = P \quad (\text{or } f_y = Q) \]

\[ \int S \, dx \]

\[ f(x, y) = \int P \, dx + g(y) \]

\[ \int \frac{dy}{dy} \]
\[ g'(y) = Q(x,y) \]
\[ g'(y) = \ldots \leftarrow \text{only } y \text{'s here, otherwise } F \text{ is not conservative.} \]
\[ \int S dy \]
\[ g(y) = \ldots + C \]
\[ \text{plug into } f(x,y) \text{ from above} \]
\[ f(x,y) = \ldots \]

II. Surfaces

• Parametrizations

\[ \vec{r}(u,v) = (\ldots, \ldots, \ldots) \quad u, v \in D \subseteq \mathbb{R}^2 \]
\[ \vec{r} \]

7) Have to specify this (often a rectangle or disk)

\[ \nabla \]

- Orientation = "up/down" = unit normal vector field on S.

\[ \vec{N} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \]

- Induced orientation on \( \partial S \)
An induced orientation on $\partial C$:
\[ \mathbf{N} \times \mathbf{T} \] points to the surface.

1. Integrals:
   - Functions
     \[
     \iint_S f \, dS = \iint_D f(\tilde{r}(u,v)) \cdot |\tilde{r}_u \times \tilde{r}_v| \, du \, dv
     \]
   - "Surface coords" (Calc 4)
   - Cartesian coords (Calc 3)
- vector fields (flux integrals)

\[ \iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S} \mathbf{F} \cdot \mathbf{N} \, dS \]

\[ = \iint_{D} \mathbf{F}(\hat{r}(u,v)) \cdot (\hat{r}_u \times \hat{r}_v) \, dudv \]

How to compute here?

- Integrals of functions
1) Integrals of functions
   Use parameterization to compute directly.

2) Integrals of vector fields
   a) Is $\mathbf{F} = \text{curl} \mathbf{G}$?
      → Apply Stoke's Theorem
      → or modify surface, but keep boundary the same
   b) Is $S$ closed (boundary of solid region)?
b) Is \( S \) closed (= boundary of solid region)?

\[ \Rightarrow \text{Apply divergence theorem.} \]

c) Compute directly

\[ \Rightarrow \text{If you know } \vec{N} \text{ and } |\vec{r}_u \times \vec{r}_v| \text{ use} \]

\[ \int_F \cdot \vec{N} \, dS = \int_S \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, du \, dv \]

\[ S \quad \text{D} \]

\[ \Rightarrow \text{Otherwise use} \]

\[ \int_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv \]

\[ D \]
Advanced trick: Add a surface to make it closed.