Curves

Parametrization
\[ \vec{r}(t), \quad a \leq t \leq b \]

Velocity/tangent vector \( \vec{r}'(t) \)

Length = \[ \int_a^b |\vec{r}'(t)| \, dt \]

Integral of a function:
\[ \int_C f \, ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}'(t)| \, dt \]

Surfaces

Parametrization
\[ \vec{r}(u,v), \quad u,v \in D \]

Vectors in the tangent plane \( \vec{r}_u, \vec{r}_v \)

Area = \[ \iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv \]

Integral of a function:
\[ \iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, du \, dv \]
Integral of a vector field \( \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \hat{T} \, ds \) along curve \( C \)

need an orientation of \( C \)

Integral of a vector field \( \int_{\Sigma} \vec{D} \cdot d\vec{S} \) over surface \( \Sigma \)

need an orientation of surfaces.

**Orientation of a surface**

For curves: choice of direction
For curves: choice of direction

$\mathbf{C}$ \quad $\mathbf{-C}$ \quad ... opposite orientation

For closed curves in the xy-plane we have a standard orientation (counter clockwise), "induced" from the plane. "positive orientation"
An orientation of a surface is a choice of a unit normal vector field.

We choose a side of the surface.

If $S = \partial E$ is the boundary of a region $E$ in $\mathbb{R}^3$, we get an "induced" orientation: we say the outward orientation is the positive orientation.
outward orientation is the positive orientation.

Example:

\[ E = \text{unit cube}, \quad \partial E = S = 6 \text{ sides} \]

\[ \cdot ) \quad E = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq 1 \right\} \quad \text{... ball} \]
\[ \partial E = S = \left\{ (x, y, z) : x^2 + y^2 + z^2 = 1 \right\} \quad \text{... sphere} \]
New feature: There are surfaces which are not orientable.

E.g. The Möbius band.

This has only one side.

Does not have two sides.
If a 2d ant crawls once around the band,
This has only one side. Once around the band, it comes back with right hand and left hand exchanged.

Another example:

Klein bottle: a closed surface (no boundary) with only one side. Must immersed itself in \( \mathbb{R}^3 \), but can be embedded without...
but can be embedded without intersection into $\mathbb{R}^4$:

Think of the 4th dimension as shades of blue (each point is $(x, y, z, b)$, where $b$ is a shade of blue).

Make most of the bottle dark blue, but the handle light blue $\rightarrow$ there is no intersection.
How to get a unit normal vector in practice?

\[ \vec{N} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \] is a unit normal (if the parametrization is smooth)

**Flux Integrals**

- \( S \) - surface
- \( \vec{F} \) - vector field

Think of \( \vec{F} \) as the velocity vector field
of a fluid flow.

\[ \text{flux} = \text{rate of flow of the fluid through the surface } S. \]

We need an orientation to decide which direction to call flow positive.

The flow through a little parallelogram is \( \vec{N} \cdot \vec{F} \text{ (area)} \).
Neg. flow

Flow parallel to the surface give no contribution

\[
\text{Flux} = \iint_S \vec{F} \cdot \vec{N} \, dS = \pm \iiint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, |\vec{r}_u \times \vec{r}_v| \, du \, dv
\]
Flux = \iint_S \vec{F} \cdot \vec{N} \, ds = \pm \iiint_D \vec{F}(\vec{r}(uv)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv

sometimes written \iint_S \vec{F} \cdot d\vec{S}

can be compared to work integral:

\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt

\text{sign depends on whether } \vec{r}_u \times \vec{r}_v \text{ is in the same direction as given orientation } \vec{N} \text{ or not.}