Homework this week is due on Friday.

Last time: parametrization of surfaces

Today: tangent planes and surface area.
Look at a point \( \vec{r}(u_0, v_0) \) of the surface:

\[ \vec{r}(u, v) \]

\[ \frac{\partial \vec{r}}{\partial u}(u_0, v_0) \]

\[ \frac{\partial \vec{r}}{\partial v}(u_0, v_0) \]

\[ \vec{r}(u_0, v_0) \]

\[ \vec{r}_u(u_0, v_0) \text{ and } \vec{r}_v(u_0, v_0) \]
the partial derivatives give us tangent vectors of the surface.

The tangent plane is the plane through the point \( \vec{r}(u_0, v_0) \) and containing the vectors \( \vec{r}_u(u_0, v_0) \) and \( \vec{r}_v(u_0, v_0) \).

**Example:** Find the tangent plane to the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1 \) at the point \( (1, 1, \frac{1}{12}) \).
1st step: find a parameterization

\[ x = 2 \sin \varphi \sin \theta \]
\[ y = 2 \sin \varphi \cos \theta \quad 0 \leq \varphi \leq \pi \]
\[ z = \cos \varphi \quad 0 \leq \theta \leq 2\pi \]

2nd step: compute partial derivatives

\[ \mathbf{r}(\varphi, \theta) = \langle 2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, \cos \varphi \rangle \]
\[ \mathbf{r}_\varphi(\varphi, \theta) = \langle 2 \cos \varphi \sin \theta, 2 \cos \varphi \cos \theta, -\sin \varphi \rangle \]
\[ \mathbf{r}_\theta(\varphi, \theta) = \langle 2 \sin \varphi \cos \theta, -2 \sin \varphi, \sin \varphi, 0 \rangle \]
What are \( \varphi, \vartheta \) to give \( (1, 1, \frac{1}{12}) \)?

\[
\begin{align*}
2 &= \frac{1}{12} = \cos \varphi \quad \Rightarrow \quad \varphi = \frac{\pi}{4} \\
x &= 1 = 2 \sin \frac{\pi}{4} \sin \vartheta \quad \Rightarrow \quad \frac{1}{12} = \sin \vartheta \\
y &= 1 = 2 \sin \frac{\pi}{4} \cos \vartheta \quad \Rightarrow \quad \frac{1}{12} = \cos \vartheta \\
&\quad \Rightarrow \quad \vartheta = \frac{\pi}{4}
\end{align*}
\]

At \( (1, 1, \frac{1}{12}) \) two tangent vectors are

\[
\vec{v}_\varphi \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \langle 1, -1, 0 \rangle, \quad \vec{v}_\vartheta \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \langle 1, 1, -\frac{1}{12} \rangle
\]

Tangent plane has parametrization:

\[
\vec{P}(u, v) = \langle 1, 1, \frac{1}{12} \rangle + u \langle 1, -1, 0 \rangle + v \langle 1, 1, -\frac{1}{12} \rangle
\]
to get an equation for the tangent plane use $\vec{N} = \langle 1, -1, 0 \rangle \times \langle 1, 1, -\frac{1}{\sqrt{2}} \rangle$ and the given point.

A parametrization is called smooth if $\vec{r}_u \times \vec{r}_v \neq 0$.

This guarantees the "coordinate grid" on the surface actually a (distorted) grid.
For example, the usual parametrization of the sphere is not smooth at the poles.

(top-down view)
Surface area:

Area in the uv-plane is different from the area of the actual surface. There is distortion!

(Compare area on a map with area on a globe.)

\[ \text{Area} = \iint \ldots \, du \, dv \]
Area = \iint_D du \, dv

Have to find a factor to account for the distortion.

Little \Delta u \times \Delta v - rectangles \mapsto \text{approximate by little parallelograms in } S

For curves

Approximation
So: \( \text{Area} (S) = \lim_{{\Delta u, \Delta v \to 0}} \sum_{i,j} |\vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j)| \Delta u \Delta v \)

\[ = \iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv \]

Example: Find the surface area of the "tropics" on a unit sphere: \( \frac{\pi}{3} \leq \varphi \leq \frac{2\pi}{3}, \quad 0 \leq \theta \leq 2\pi \)
... Next time